KOLMOGOROV'S ZERO-ONE LAW WITH APPLICATIONS

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Abstract. This expository paper requires little previous background in rigorous probability theory. In order to provide a comprehensive understanding, we begin with the foundations of probability theory, building to a proof of Kolmogorov's Zero-One Law. Following the proof, we examine applications of this law in other areas of mathematics, namely percolation theory and random power series.

Contents

1. Introduction

The central focus of this paper is Kolmogorov's Zero-One Law. This theorem states that given a sequence of independent events $A_1, A_2, \dots$ in a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, events residing in the tail field of this sequence occur with a probability of either zero or one.

After proving Kolmogorov's Zero-One Law, some applications are explored. Firstly, we examine the presence of an infinite connected component in randomly generated subgraphs of $\mathbb{Z}^2$, the coordinate plane. We find that depending on the percolation parameter p, an infinite connected component occurs with probability zero or one. Following this, we state that the critical parameter is $p = \frac{1}{2}$, i.e. an infinite component occurs with probability one if $p \geq \frac{1}{2}$, and zero if $p < \frac{1}{2}$.

After stating an alternative formulation of Kolmogorov's Zero-One Law using random variables, we examine the convergence of random power series, i.e. series of the form

$$\sum_{n=0}^{\infty} X_n(\omega) z^n$$

Date: October 30, 2017.

where $X_0, X_1, X_2, \dots$ are random variables. We find that the radius of convergence of such a series is constant with probability one.

2. Preliminaries

Definition 2.1. Given a set Ω, a *semialgebra* $\mathcal{J}$ of Ω is a collection of subsets of Ω with the following properties:

- $\Omega, \emptyset \in \mathcal{J}$;
- $\mathcal{J}$ is closed under finite intersection; and
- The complement of any element in $\mathcal{J}$ can be expressed as the finite disjoint union of elements of $\mathcal{J}$.

Explicitly, the last bullet says that for any $A \in \mathcal{J}$, there exists $n \in \mathbb{N}$ and $B_1, \dots, B_n \in \mathcal{J}$ such that

$$A^C = B_1 \,\dot{\cup}\, B_2 \,\dot{\cup} \cdots \dot{\cup}\, B_n, \tag{2.2}$$

where we use $\dot{\cup}$ to denote the union of two disjoint sets.

Definition 2.3. Given a set Ω, an *algebra* or *field of sets* $\mathcal{C}$ is a collection of subsets of Ω with the following properties:

- $\Omega, \emptyset \in \mathcal{C}$; and
- $\mathcal{C}$ is closed under finite unions, intersections, and complements.

Thus, given any finite number of subsets $A_1, A_2, \dots, A_n \in \mathcal{C}$,

$$A_1^C \ \in \ \mathcal{C}; \tag{2.4}$$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n \ \in \ \mathcal{C}; \tag{2.5}$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n \ \in \ \mathcal{C}. \tag{2.6}$$

Example 2.7. Algebras are often quite useful, but in many cases closure under finite operations is lacking. Let $\Omega = \mathbb{N}$, and consider the collection of all finite subsets of $\mathbb{N}$ and their complements. This is an algebra. Yet restriction to finitely many operations precludes important subsets of $\mathbb{N}$ like $2\mathbb{N}$ and $2\mathbb{N}+1$, the sets of even and odd natural numbers, respectively, from being in the algebra.

Any algebra over a set Ω may still be lacking many important subsets of Ω as above. What we need to avoid these types of problems is closure under *countably* many operations.

Definition 2.8. Given a set Ω, a *σ-algebra* (pronounced "sigma algebra") $\mathcal{F}$ is a collection of subsets of Ω with the following properties:

- $\Omega, \emptyset \in \mathcal{F}$; and
- $\mathcal{F}$ is closed under countable unions, intersections, and complements.

Thus, given countably many elements $A_1, A_2, A_3, \dots \in \mathcal{F}$,

$$A_1^C \in \mathcal{F} \tag{2.9}$$

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots \in \mathcal{F}; \tag{2.10}$$

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \cdots \in \mathcal{F}. \tag{2.11}$$

Clearly, all σ-algebras are algebras and all algebras are semialgebras. The most important and powerful of these is the σ-algebra, which will be necessary for us to consider the concept of infinity, with respect to probability. It is important to note that none of these collections above will necessarily contain *all* subsets of Ω, although 2^Ω is an example of a σ-algebra.

Given $\mathcal{A}$ a collection of subsets of Ω, not necessarily closed under any operations, we may talk about the smallest σ-algebra containing $\mathcal{A}$. We denote this by $\sigma(\mathcal{A})$ and say it is the "σ-algebra generated by $\mathcal{A}$" [1].

For any collection of subsets $\mathcal{A}$, $\sigma(\mathcal{A})$ is well-defined. We define it as the intersection of all σ-algebras containing $\mathcal{A}$; this is indeed a σ-algebra, as intersections of σ-algebras with other σ-algebras yields a σ-algebra. As 2^Ω contains $\mathcal{A}$ and is itself a σ-algebra, we have existence. For uniqueness, suppose $\sigma(\mathcal{A}) = S_1$ and $\sigma(\mathcal{A}) = S_2$, with $S_1 \neq S_2$. As S_1 is the intersection of all σ-algebras containing $\mathcal{A}$, and S_2 contains $\mathcal{A}$, we have $S_1 \subseteq S_2$. By the same logic, $S_2 \subseteq S_1$, so clearly $S_1 = S_2$.

For an example, consider the definition below.

Definition 2.12. The *Borel subsets of* $\mathbb{R}$, which we shall denote $\mathcal{B}$, are defined as follows. If we let

$$\mathcal{I} = \{\text{open intervals of } \mathbb{R}\} \tag{2.13}$$

then $\mathcal{B} = \sigma(\mathcal{I})$.

So the sets that can be formed by countably many set operations on open sets are the Borel sets (this includes closed sets, singleton elements, half open or half closed sets, etc.). Just about any subset of $\mathbb{R}$ one could name is a Borel set [1]. For example, $\mathbb{Q}$ is Borel. Because the rational numbers are countable, we can write:

$$\mathbb{Q} = \bigcup_{q\in\mathbb{Q}} [(-\infty, q) \cup (q, \infty)]^C. \tag{2.14}$$

There are some further facts that we will make use of in proofs that follow.

Theorem 2.15. *(De Morgan's Laws) Given a collection of subsets $\{A_\alpha\}_{\alpha\in I}$ we have the following properties of intersections, unions, and complements:*

$$\left(\bigcup_\alpha A_\alpha\right)^C = \bigcap_\alpha A_\alpha^C; \tag{2.16}$$

$$\left(\bigcap_\alpha A_\alpha\right)^C = \bigcup_\alpha A_\alpha^C. \tag{2.17}$$

Proof. We examine 2.16 first. Suppose $x \in (\bigcup_\alpha A_\alpha)^C$. This is equivalent to saying x is in none of the sets. So it must reside in the complement of each set, i.e. $x \in \bigcap_\alpha A_\alpha^C$. This shows $(\bigcup_\alpha A_\alpha)^C \subseteq \bigcap_\alpha A_\alpha^C$. For the other direction, suppose $x \in \bigcap_\alpha A_\alpha^C$. This is equivalent to saying x is in none of the individual sets. So clearly x is not in the union of these sets, i.e. $x \in (\bigcup_\alpha A_\alpha)^C$. So $\bigcap_\alpha A_\alpha^C \subseteq (\bigcup_\alpha A_\alpha)^C$.

Having shown both directions, we have $\bigcap_\alpha A_\alpha^C = (\bigcup_\alpha A_\alpha)^C$. The proof for 2.17 is virtually identical. □

3. Probability Essentials

Definition 3.1. A *probability triple* or *probability space* is a triple $(\Omega, \mathcal{F}, \mathbf{P})$ consisting of:

- The *sample space* Ω;
- $\mathcal{F}$, a σ-algebra of Ω; and
- The *probability measure* $\mathbf{P}$, a function $\mathbf{P} : \mathcal{F} \to [0,1]$, such that $\mathbf{P}(\emptyset) = 0$, $\mathbf{P}(\Omega) = 1$, and $\mathbf{P}$ is countably additive on disjoint sets, i.e.

$$\mathbf{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbf{P}(A_1) + \mathbf{P}(A_2) + \cdots = \sum_{i=1}^{\infty} \mathbf{P}(A_i) \tag{3.2}$$

  where all the A_i are disjoint.

The elements of $\mathcal{F}$ are those subsets of Ω which have well-defined probabilities. We call these *events*, and for any $A \in \mathcal{F}$, we speak of $\mathbf{P}(A)$ as the "probability of event A" [1].

Remark. The nature of the probability measure matches our intuition from everyday life. Consider a spinner on a circle with four different colored and equally sized quadrants: red, blue, green, and yellow. The probability of landing on each is $\frac{1}{4}$. The fact that each of the colors is disjoint (i.e. non-overlapping) means that we ought to add the probabilities when we combine them, e.g. $\mathbf{P}(\text{red or yellow}) = \mathbf{P}(\text{red}) + \mathbf{P}(\text{yellow}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. This makes perfect sense, because we are now asking for the probability that we land on exactly half of the circle, namely the half consisting of the red and yellow quadrants.

A situation such as this makes clear also why $\mathbf{P}(\Omega) = 1$. In this situation, Ω is the whole circle. Every time we spin, the spinner must land somewhere on the circle. Evaluating $\mathbf{P}(\Omega)$ is equivalent then to asking the question: "what is the probability the spinner lands on the circle?" The probability must be one, and we accordingly define all probability measures to have $\mathbf{P}(\Omega) = 1$.

Remark. The reader may ask: is the structure of the σ-algebra really necessary for the definition of the probability measure? We attempt to motivate this by showing how the definition falls apart if we try to construct uncountable additivity. Let $\Omega = [0,1]$, and let $\mathbf{P}$ be Lebesgue measure on $[0,1]$; for the unfamiliar reader, Lebesgue measure gives the "length" of any interval, i.e. $\mathbf{P}((a,b)) = \mathbf{P}((a,b]) = \mathbf{P}([a,b)) = \mathbf{P}([a,b]) = b - a$. Now consider

$$\bigcup_{x \in [0,1]} \{x\} = [0,1] = \Omega. \tag{3.3}$$

We have an uncountable disjoint union of the singleton sets. Let's try to apply additivity over disjoint sets to this union.

$$\mathbf{P}\left(\bigcup_{x \in [0,1]} \{x\}\right) = \sum_{x \in [0,1]} \mathbf{P}(\{x\}) = \sum_{x \in [0,1]} 0 = 0 \tag{3.4}$$

because the singleton sets have Lebesgue measure zero. Yet we should have by definition of the probability measure that

$$\mathbf{P}\Bigg(\bigcup_{x\in[0,1]}\{x\}\Bigg) = \mathbf{P}(\Omega) = 1, \tag{3.5}$$

a clear contradiction. An example such as this shows that we cannot permit uncountable additivity and still have our probability measure behave as one would expect it to. As we still desire countable additivity, the next best option, the σ-algebra is the most logical thing on which to define the probability measure [1].

Remark. The reader may ask also: what is the necessity of defining other σ-algebras when we have the power set? Could we not always define $\mathbf{P} : 2^{\Omega} \to [0,1]$? The answer is that in many instances the probability measure $\mathbf{P}$ may not be well-defined on *every* subset of Ω. For example, Rosenthal spends all of chapter one proving that on the sample space $[0,1]$ there does not exist a probability measure defined on $2^{[0,1]}$ that gives the length of every interval [1]. This is because certain subsets of $[0,1]$, if carefully constructed, lead to contradictions in the probability measure (e.g. Vitali sets), and cannot be assigned a "length" in any reasonable sense [3] [2].

There are a few useful properties of the probability measure we point out for use in future proofs.

Lemma 3.6. *The probability measure demonstrates monotonicity, i.e. if $A \subseteq B$, then $\boldsymbol{P}(A) \leq \boldsymbol{P}(B)$.*

Proof. Let $A \subseteq B$. Then $B = A \mathbin{\dot\cup} (B \cap A^C)$. These two sets are disjoint, so by countable additivity it follows that

$$\mathbf{P}(B) = \mathbf{P}(A) + \mathbf{P}(B \cap A^C) \geq \mathbf{P}(A). \tag{3.7}$$

□

Although countable additivity applies only to disjoint sets, we also have *countable subadditivity* of $\mathbf{P}$ over all sets, defined as below.

Lemma 3.8. *Given $A_1, A_2, \ldots \in \mathcal{F}$, not necessarily disjoint, we always have:*

$$\boldsymbol{P}\Bigg(\bigcup_{i=1}^{\infty} A_i\Bigg) \leq \sum_{i=1}^{\infty} \boldsymbol{P}(A_i). \tag{3.9}$$

Proof. Indeed, we rewrite the A_i as disjoint sets as follows:

$$\mathbf{P}(A_1 \cup A_2 \cup A_3 \cup \cdots) = \mathbf{P}(A_1 \mathbin{\dot\cup} (A_2 \cap A_1^C) \mathbin{\dot\cup} (A_3 \cap A_2^C \cap A_1^C) \mathbin{\dot\cup} \cdots). \tag{3.10}$$

From here we may apply countable additivity to get

$$= \mathbf{P}(A_1) + \mathbf{P}(A_2 \cap A_1^C) + \mathbf{P}(A_3 \cap A_2^C \cap A_1^C) + \cdots \leq \mathbf{P}(A_1) + \mathbf{P}(A_2) + \mathbf{P}(A_3) \cdots \tag{3.11}$$

where we use the fact that $A_n \cap A_{n-1}^C \cap \cdots \cap A_1^C \subseteq A_n$ and apply monotonicity in the last inequality. □

Remark. By definition we immediately have $1 = \mathbf{P}(\Omega) = \mathbf{P}(A \mathbin{\dot\cup} A^C) = \mathbf{P}(A) + \mathbf{P}(A^C)$, implying that for any event A, $\mathbf{P}(A^C) = 1 - \mathbf{P}(A)$.

How often do probability triples exist? Its criteria may seem difficult to meet, but in fact we can often construct probability triples quite easily with the help of the following immense theorem.

Theorem 3.12. *(The Extension Theorem) Let $\mathcal{S}$ be a semialgebra of subsets of Ω, and $\boldsymbol{P}$ a function $\boldsymbol{P} : \mathcal{S} \to [0,1]$ such that $\boldsymbol{P}(\emptyset) = 0$ and $\boldsymbol{P}(\Omega) = 1$. Suppose $\boldsymbol{P}$ satisfies finite additivity over disjoint sets, i.e.*

$$\boldsymbol{P}(A_1 \dot\cup \cdots \dot\cup A_t) = \sum_{i=1}^{t} \boldsymbol{P}(A_i) \text{ for disjoint } A_1, \ldots, A_t \in \mathcal{S}, \ \bigcup_{i=1}^{t} A_i \in \mathcal{S}, \tag{3.13}$$

and that $\boldsymbol{P}$ also satisfies countable subadditivity, i.e.

$$\boldsymbol{P}\left(\bigcup_{i=1}^{\infty} A_i \right) \leq \sum_{i=1}^{\infty} \boldsymbol{P}(A_i) \text{ for countably many } A_i \in \mathcal{S} \text{ if the union is in } \mathcal{S}. \tag{3.14}$$

Then there exists a σ-algebra $\mathcal{M}$ and a probability measure $\boldsymbol{P}^*$ defined on $\mathcal{M}$, such that $\mathcal{S} \subseteq \mathcal{M}$, $\boldsymbol{P}^*(A) = \boldsymbol{P}(A)$ for any $A \in \mathcal{S}$, and $(\Omega, \mathcal{M}, \boldsymbol{P}^*)$ is a valid probability triple.

Though we omit the proof because of its length, one should not neglect the importance of this theorem. Given only a semialgebra and a probability measure satisfying finite additivity and countable subadditivity, we are *guaranteed* a valid probability triple. We include a statement of the theorem so that the phrase "given a probability triple $(\Omega, \mathcal{F}, \mathbf{P})$", which we shall use often from here onward, carries some meaning.

Definition 3.15. Given a probability triple $(\Omega, \mathcal{F}, \mathbf{P})$, a collection of events (possibly infinite) $\{E_\alpha\}_{\alpha \in A}$ is *independent* if for all $n \in \mathbb{N}$, and all possible finite combinations $\alpha_1, \alpha_2, \ldots, \alpha_n$, we have:

$$\mathbf{P}(E_{\alpha_1} \cap E_{\alpha_2} \cap \cdots \cap E_{\alpha_n}) = \mathbf{P}(E_{\alpha_1})\mathbf{P}(E_{\alpha_2}) \cdots \mathbf{P}(E_{\alpha_n}). \tag{3.16}$$

If our collection contains only two events A and B, a situation perhaps more familiar, then clearly independence tells us that $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$. Note that pairwise independence is not sufficient; we need every possible finite subcollection to be independent for the entire collection to be independent [1].

Example 3.17. To see that pairwise independence does not imply independence, consider one roll of a fair 9-sided die. Let A be the event we roll a 1, 2 or 3; B be the event we roll a 3, 4, or 5; C be the event we roll a 5, 6, or 1. Then we have the following:

$$\mathbf{P}(A) = \mathbf{P}(B) = \mathbf{P}(C) = \frac{1}{3}; \tag{3.18}$$

$$\mathbf{P}(A \cap B) = P(\{3\}) = \frac{1}{9} = \mathbf{P}(A)\mathbf{P}(B); \tag{3.19}$$

$$\mathbf{P}(B \cap C) = P(\{5\}) = \frac{1}{9} = \mathbf{P}(B)\mathbf{P}(C); \tag{3.20}$$

$$\mathbf{P}(A \cap C) = P(\{1\}) = \frac{1}{9} = \mathbf{P}(A)\mathbf{P}(C). \tag{3.21}$$

In other words, we have A, B, and C pairwise independent. This tells us that given any one of the three, we are just as likely to have either of the other two. But clearly $\mathbf{P}(A \cap B \cap C) = 0 \neq \frac{1}{27} = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C)$. So the events are not independent. If we have A and B, for example, then we do not have C by necessity. The occurrence of C is clearly dependent on A and B; the dependencies of these events among each other demonstrate that they are not independent.

4. Moving Towards Infinity

Definition 4.1. Given a probability triple $(\Omega, \mathcal{F}, \mathbf{P})$ and infinitely many events $A_1, A_2, A_3, \ldots \in \mathcal{F}$, define a new event $\{A_n \text{ i.o.}\} \in \mathcal{F}$, read "$A_n$ infinitely often," as

$$\{A_n \text{ i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k, \tag{4.2}$$

and another event $\{A_n \text{ a.a.}\} \in \mathcal{F}$, read "$A_n$ almost always", as

$$\{A_n \text{ a.a}\} = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k. \tag{4.3}$$

Regarding each A_n as a subset of the set Ω, "A_n infinitely often" is the set of all $\omega \in \Omega$ which are in infinitely many of the A_n. "A_n almost always" is more complex to interpret. If $\omega \in \Omega$ is in $\{A_n \text{ a.a.}\}$, then there exists $m_\omega \in \mathbb{N}$ such that

$$\forall\ n > m_\omega,\ \omega \in A_n,\ \text{ i.e. } \omega \in \bigcap_{i=m_\omega+1}^{\infty} A_i. \tag{4.4}$$

So $\omega \in \{A_n \text{ a.a.}\}$ implies that ω is in all but a finite number of the A_n. Colloquially, then, we say $\{A_n \text{ a.a.}\}$ is the event that all but finitely many of the events A_n occur.

Lastly, we should note that since $A_1, A_2, \ldots \in \mathcal{F}$, the closure of the σ-algebra under countable operations ensures that $\{A_n \text{ i.o.}\}$, $\{A_n \text{ a.a.}\} \in \mathcal{F}$, and hence are events in their own right with well defined probabilities [1].

Example 4.5. Consider a countable number of fair die rolls with the standard six sided die. Let our events be $S_1, S_2, S_3, \ldots$, where S_i is the event that the i^{th} roll is a 6. In this situation, $\{S_n \text{ i.o.}\}$ is the event that we roll infinitely many 6's. Then $\{S_n \text{ a.a.}\}$ is the event that we roll a 6 all but finitely many times, i.e. only a finite number of 1's, 2's, 3's, 4's, or 5's. An example like this should make clear that "almost always" is stronger than "infinitely often." The former implies the latter, but not vice versa, e.g. we could easily have both infinitely many 6's and 5's in our result.

Theorem 4.6. *(The Borel-Cantelli Lemma) Let $A_1, A_2, \ldots \in \mathcal{F}$.*

$$\textit{(i) If } \sum_n \boldsymbol{P}(A_n) < \infty, \textit{then } \boldsymbol{P}(\{A_n \textit{ i.o.}\}) = 0.$$

$$\textit{(ii) If } \sum_n \boldsymbol{P}(A_n) = \infty, \{A_n\}_{n=1}^{\infty} \textit{ independent, then } \boldsymbol{P}(\{A_n \textit{ i.o.}\}) = 1.$$

Proof. Let's start with i). Note that for all $m \in \mathbb{N}$, we have:

$$\mathbf{P}(\{A_n \text{ i.o.}\}) = \mathbf{P}\bigg(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k\bigg) \leq \mathbf{P}\bigg(\bigcup_{k=m}^{\infty} A_k\bigg)$$

by monotonicity. Then it follows from countable subadditivity of the probability measure that

$$\mathbf{P}\bigg(\bigcup_{k=m}^{\infty} A_k\bigg) \leq \sum_{k=m}^{\infty} \mathbf{P}(A_k).$$

Convergence of this last series implies that the terms in the summation go to zero. Hence for all $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that $\sum_{k=m}^{\infty} \mathbf{P}(A_k) < \epsilon$. It follows that $\mathbf{P}(A_n \text{ i.o.}) < \epsilon$ for any ϵ, hence it is 0.

Now for ii). Because $\mathbf{P}(\{A_n \text{ i.o.}\}) = 1 - \mathbf{P}(\{A_n \text{ i.o.}\}^C)$, it suffices to show that $\mathbf{P}(\{A_n \text{ i.o.}\}^C) = 0$. By De Morgan's Laws in the preliminaries we have:

$$\mathbf{P}(\{A_n \text{ i.o.}\}^C) = \mathbf{P}\Bigg(\Bigg(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\Bigg)^C\Bigg) = \mathbf{P}\Bigg(\bigcup_{n=1}^{\infty}\Bigg(\bigcup_{k=n}^{\infty} A_k\Bigg)^C\Bigg) = \mathbf{P}\Bigg(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k^C\Bigg).$$

Then we can write by countable subadditivity that

$$\mathbf{P}\Bigg(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k^C\Bigg) \leq \sum_{n=1}^{\infty} \mathbf{P}\Bigg(\bigcap_{k=n}^{\infty} A_k^C\Bigg).$$

So we need only show that for all $n \in \mathbb{N}$, $\mathbf{P}(\bigcap_{k=n}^{\infty} A_k^C) = 0$. By monotonicity we have that for all $m \in \mathbb{N}$,

$$\mathbf{P}\Bigg(\bigcap_{k=n}^{\infty} A_k^C\Bigg) \leq \mathbf{P}\Bigg(\bigcap_{k=n}^{n+m} A_k^C\Bigg) = \prod_{i=1}^{n+m} \mathbf{P}(A_k^C),$$

where the second step follows by definition of independent events, and the fact that the complements of independent events are also independent. Now using the fact that for all $x \in \mathbb{R}$, $1 - x \leq e^{-x}$, and letting $\mathbf{P}(A_k^C) = 1 - \mathbf{P}(A_k)$, we have:

$$\prod_{i=1}^{n+m} 1 - \mathbf{P}(A_k) \leq \prod_{i=1}^{n+m} e^{-\mathbf{P}(A_k)} = e^{-\sum_{i=1}^{n+m} \mathbf{P}(A_k)}.$$

Because the sum diverges to infinity, the last term goes to zero as $m \to \infty$. It follows that $\mathbf{P}(\bigcap_{k=n}^{\infty} A_k^C)) < \epsilon$ for any ϵ, so it is 0. $\square$

The lemma tells us that if the events $\{A_n\}$ are independent, then $\mathbf{P}(\{A_n \text{ i.o.}\})$ is either 0 or 1 and nothing else. The reader should consider this lemma a very weak version of a zero-one law [1].

Example 4.7. Consider an infinite heavily weighted coin tossing. Let our independent events be $H_1, H_2, H_3, \ldots$, where H_i is the event that the i^{th} coin is heads. Suppose also that our coins are heavily weighted *against* flipping heads, with $\mathbf{P}(H_n) = \frac{1}{n}$ (e.g. the millionth coin has only a one in a million chance at being heads). The Borel-Cantelli Lemma tells us nevertheless that we will still flip infinitely many heads, i.e. $\mathbf{P}(\{H_n \text{ i.o.}\}) = 1$. This follows from the divergence of the harmonic series, since $\sum_n \mathbf{P}(H_n) = \sum_n \frac{1}{n}$. Such a result is not at all obvious without the lemma.

For an even less obvious result, consider an infinite coin tossing heavily weighted *in favor* of heads. Again let our events be $H_1, H_2, H_3, \ldots$, and suppose that for all $n \in \mathbb{N}$, $\mathbf{P}(H_n) = (\frac{99}{100})^n$. In other words, there is a 99% chance the first coin is heads, a 98.01% chance that the second one is heads, etc. In this scenario, $\mathbf{P}(\{H_n \text{ i.o}\}) = 0$; we cannot have infinitely many heads. This follows from the fact that

$$\sum_n \mathbf{P}(H_n) = \sum_n \left(\frac{99}{100}\right)^n = \frac{\frac{99}{100}}{1 - \frac{99}{100}} = 99 < \infty \tag{4.8}$$

because this forms a geometric series with common ratio $r = \frac{99}{100}$. This result is entirely unintuitive, but revealing of the great power of the lemma.

Example 4.9. As a rather amusing example, consider the event that a monkey typing at random would produce Shakespeare's *Hamlet* in an infinite amount of time. Let's ignore case sensitivity, but otherwise we still expect our monkey to type not only all letters, but spaces, quotes, commas, periods, and other punctuation correctly. To be safe, let's assume there are 45 possible characters. Moreover, let's give the monkey an old fashioned typewriter with no delete key so we need not worry about backspaces. *Hamlet* has some finite number of characters N, with N large. Now consider the infinite string produced by our monkey typing at random. We assume for simplicity that each character has the same probability of being hit and that hits are independent. We seek a substring that is the text of *Hamlet.* If we pick an arbitrary starting point in the infinite string the probability that this is the beginning of a full text of *Hamlet* is:

$$\mathbf{P}(H) = \left(\frac{1}{45}\right)^N = \epsilon > 0. \tag{4.10}$$

Now consider a sequence of events $S_1, S_{N+1}, S_{2N+1}, \ldots$, where S_i is the event that the i^{th} character is the start of a full text of *Hamlet.* These events are independent because they specify the start of a *Hamlet*-length substring of our infinite string with no overlap. Clearly then $\mathbf{P}(S_i) = \epsilon$ always. It follows that

$$\sum_{i=1}^{\infty} \mathbf{P}(S_{2i+1}) = \sum_{i=1}^{\infty} \epsilon = \infty. \tag{4.11}$$

So by Borel-Cantelli, $\mathbf{P}(\{S_i \text{ i.o.}\}) = 1$. In other words, our monkey will not only type *Hamlet*, but will do so infinitely many times.

Definition 4.12. Given a sequence of events $A_1, A_2, \ldots \in \mathcal{F}$, we define their *tail field* as

$$\tau = \bigcap_{n=1}^{\infty} \sigma(A_n, A_{n+1}, A_{n+2}, \ldots). \tag{4.13}$$

The tail field is a σ-algebra whose members we call *tail events.*

The tail field has some interesting attributes in the case that the A_i are independent. Then any $T \in \tau$ cannot depend on any particular event A_i, or on any finite number of events $A_{n_1}, A_{n_2}, \ldots, A_{n_m}$, $n_i \in \mathbb{N}$. If n_{max} is the highest index, then none of these are in $\sigma(A_{n_{max}+1}, A_{n_{max}+2}, \ldots)$ and hence are not in the tail field by Lemma 5.1 (see below), so bear no relation whatsoever to T. All tail events clearly depend strongly on the tail of our sequence of events; events of this nature depend on infinitely many A_i. As an easy example, $\{A_n \text{ i.o.}\}, \{A_n \text{ a.a.}\} \in \tau$. While the Borel-Cantelli Lemma can only be applied to the events stated, Kolmogorov's Zero-One Law is more powerful in that it applies to any tail event [1].

5. Kolmogorov's Zero-One Law

First a few lemmas on independence that we'll need to make use of.

Lemma 5.1. *Let $B_1, B_2, B_3 \ldots$ be independent. Then $\sigma(B_1, \ldots, B_{i-1}, B_{i+1}, \ldots)$ and $\sigma(B_i)$ are independent classes, i.e. for all $X \in \sigma(B_1, \ldots, B_{i-1}, B_{i+1}, \ldots)$, $\boldsymbol{P}(B_i \cap X) = \boldsymbol{P}(B_i)\boldsymbol{P}(X)$.*

The proof, which we omit, is quite lengthy and involves some technical results of the extension theorem, but the result should not be surprising. If we have events that we already know are independent from B_i, then the lemma simply says that performing countably many set operations on these yields a result that is also independent from B_i.

We use this lemma to help prove another.

Lemma 5.2. *Let $A_1, A_2, \dots, B_1, B_2, \dots$ be a collection of independent events.*

(i) If $X \in \sigma(A_1, A_2, \dots)$, then $X, B_1, B_2, \dots$ are independent.

(ii) The σ-algebras $\sigma(A_1, A_2, \dots)$ and $\sigma(B_1, B_2, \dots)$ are independent classes, i.e. if $X \in \sigma(A_1, A_2, \dots), Y \in \sigma(B_1, B_2, \dots)$, then $\boldsymbol{P}(X \cap Y) = \boldsymbol{P}(X)\boldsymbol{P}(Y)$.

Proof. Consider some arbitrary finite set of indices, $i_1, i_2, \dots, i_n$, and let $F = B_{i_1} \cap B_{i_2} \cap \dots \cap B_{i_n}$. Then clearly $F, A_1, A_2, \dots$ are independent because all of the B's were included in the original independent set. By Lemma 5.1, for any $X \in \sigma(A_1, A_2, \dots)$, we have F, X independent, and $\mathbf{P}(F \cap X) = \mathbf{P}(F)\mathbf{P}(X)$. Because F is arbitrary and for any set of indices this is true, it follows that $X, B_1, B_2, \dots$ is an independent collection, because the definition of independence simply requires that any finite subcollection is independent for the entire set to be. This proves (i). Applying Lemma 5.1 once more, and picking any $Y \in \sigma(B_1, B_2, \dots)$, it follows that X and Y are independent and $\mathbf{P}(X \cap Y) = \mathbf{P}(X)\mathbf{P}(Y)$. This proves (ii). $\square$

Finally, we are ready to prove Kolmogorov's Zero-One Law.

Theorem 5.3. *(Kolmogorov's Zero-One Law) Given a probability triple $(\Omega, \mathcal{F}, \boldsymbol{P})$ and a sequence of independent events $A_1, A_2, \dots \in \mathcal{F}$ with tail field τ, if $T \in \tau$, then $\boldsymbol{P}(T) \in \{0, 1\}$.*

Proof. We have an independent collection $A_1, A_2, A_3, \dots$, and our tail event $T \in \tau$. Then for any $n \in \mathbb{N}$, as $T \in \sigma(A_{n+1}, A_{n+2}, \dots)$, we have $T, A_1, A_2, \dots, A_n$ independent by Lemma 5.2 (i). It follows that $T, A_1, A_2, \dots$ is an independent collection. If we pick any finite subcollection with indices $m_1, m_2, \dots m_k$, with m_{max} the largest of these, we need only let $n > m_{max}$ to automatically have T independent from $A_{m_1}, \dots, A_{m_k}$ by above.

So, with $T, A_1, A_2, \dots$ independent, by Lemma 5.1 we then have T and S independent for any $S \in \sigma(A_1, A_2, \dots)$. But we also know by definition that $T \in \tau \subseteq \sigma(A_1, A_2, \dots)$, i.e. T is independent of itself! It follows that

$$\mathbf{P}(T) = \mathbf{P}(T \cap T) = \mathbf{P}(T)\mathbf{P}(T) = \mathbf{P}(T)^2,$$

so $\mathbf{P}(T) = 0$ or $\mathbf{P}(T) = 1$. $\square$

Example 5.4. Let a_n be a fixed sequence of real numbers. Consider the space $\Omega = \{\pm 1\}^\infty$. Consider a *sign sequence* $\omega_n \in \Omega$, such that $\mathbf{P}(\{\omega_i = 1\}) = \frac{1}{2}$ and $\mathbf{P}(\{\omega_i = -1\}) = \frac{1}{2}$, where ω_i is the i^{th} component of the infinite sequence ω_n. Then the event that $\sum_n \omega_n a_n$ converges is a tail event. If the sum converges, then changing the signs of finitely many terms will yield a sum that must also converge; and similarly if it diverges, changing the signs of finitely many terms will still yield a divergent series. As no finite number of sign switches can change the convergence, then the event that the series converges resides in the tail field. Hence, by Kolmogorov's Zero-One Law,

$$\mathbf{P}\bigg(\sum_n \omega_n a_n \text{ converges}\bigg) \in \{0,1\}. \tag{5.5}$$

6. Application to Percolation Theory

We assume the reader has some background in graph theory. *Percolation* of graphs examines the connectedness of randomly generated graphs.

Definition 6.1. Given a graph $G = (V, E)$, a *subgraph* of G, denoted G', is any graph $G' = (V', E')$ where $E' \subseteq E$ and $V' \subseteq V$.

Bond percolation examines randomly generated subgraphs where $V' = V$. In other words, only edges are lost while all vertices remain. For the remainder of this paper we will exclusively consider subgraphs of this sort [6].

Definition 6.2. Let $G = (V, E)$. In constructing a random subgraph G', we define the *bond percolation parameter* $p \in [0,1]$ as the probability of edge retention, independent of edge. That is, for all $e \in E$,

- $\mathbf{P}(e \in E') = p$
- $\mathbf{P}(e \notin E') = 1 - p$.

We say retained edges are *open* and removed edges *closed*. It will be convenient to denote open edges with 1's and closed edges with 0's [4].

Consider bond percolation on $\mathbb{Z}^2 = (V, E)$, the coordinate plane.

Question 6.3. *Given a randomly generated subgraph, what is the probability that there exists an infinite cluster, i.e. an infinite connected component of non-repeating open edges?*

We wish to apply Kolmogorov's Zero-One Law, but we first need a valid probability triple to work in.

With $\mathbb{Z}^2 = (V, E)$, first fix an ordering of the edge set E, i.e. $E = \{e_1, e_2, \dots\}$. Let $\Omega = \prod_{e \in E}\{0,1\}$ (where $\prod$ is the Cartesian product), the space of all infinite binary sequences. Then every $\omega \in \Omega$, which we shall call a *configuration*, specifies a particular subgraph of $\mathbb{Z}^2$ [7]. Letting ω_i denote the i^{th} component of ω, for a given configuration we know edge e_i is open if $\omega_i = 1$, and closed if $\omega_i = 0$.

Now we need a σ-algebra of subsets of Ω. For arbitrary $n \in \mathbb{N}$, let $f \in \mathbb{F}_2^n$ be a vector that specifies finitely many 0's and 1's, and let f_i denote the i^{th} component, $1 \le i \le n$. Now, for all $n \in \mathbb{N}$, we consider the cylinder sets of Ω, i.e. the sets of the form:

$$S_f = \{\omega \in \Omega : \omega_i = f_i\}. \tag{6.4}$$

In words, a cylinder set specifies the states of the first n edges with a fixed combination of 0's and 1's; to reside in the set, ω must be match these on the first n edges, but can have random assignment thereafter. Our σ-algebra, denoted $\mathcal{F}$, will then be generated by *all* possible cylinder sets, i.e. over all $n \in \mathbb{N}$ and $f \in \mathbb{F}_2^n$ [5].

We need a valid probability measure defined on $\mathcal{F}$. Given a bond percolation parameter $p \in [0,1]$, we first define $q : \{0,1\} \to [0,1]$ (note the difference between set and interval notation) with the following properties:

- $q(\{0,1\}) = 1$;
- $q(\emptyset) = 0$;
- $q(1) = p$;
- $q(0) = 1 - p$.

This is a valid probability measure on $\{0, 1\}$. Though beyond the scope of this paper, it is true that the product of probability measures is indeed a probability measure. This is called a *product measure*. So on our space $\Omega = \prod_{e \in E}\{0, 1\}$, we define our probability measure $\mathbf{P} = \prod_{e \in E} q$ [4].

It suffices to show that the event $C_\infty :=$ {there exists an infinite cluster} resides in the tail field in order to apply Kolmogorov's Zero-One Law. We already know that the presence of individual edges is independent, the only other necessary condition. Similar to above, we define σ_n as the the σ-algebra generated by the cylinder sets about $f \in \mathbb{F}_2^n$. The only difference is that this time we pick specific n, instead of considering all $n \in \mathbb{N}$ together as we did for $\mathcal{F}$. For any n, there are 2^n cylinder sets because there are 2^n options for $f \in \mathbb{F}_2^n$, and σ_n will be the σ-algebra generated by these 2^n sets.

Then our tail field is

$$\tau = \bigcap_{n=1}^{\infty} \sigma_n.$$

C_∞ is in the tail field. The existence of an infinite cluster could not possibly depend on any finite number of edges; indeed, we can ignore any finite number of them and this will not affect the existence of an infinite cluster. So it follows that for any n, $C_\infty \in \sigma_{n+1}$, where n edges are fixed. This is equivalent to saying $C_\infty \in \tau$ [6].

Then by Kolmogorov's Zero-One Law, $\mathbf{P}_p(C_\infty) \in \{0, 1\}$. Kolmogorov taunts us though, in that his law both gives and withholds much information. We know the probability is either 0 or 1, but by what means are we to determine which? This question is often much more difficult to answer than one would think. It is beyond the scope of this paper to show, but there is a solution [6].

Theorem 6.5. *On the lattice $\mathbb{Z}^2$, the critical percolation parameter is $p_c = \frac{1}{2}$ and we have the following:*

- *if $p < \frac{1}{2}$, then $\boldsymbol{P}(C_\infty) = 0$;*
- *if $p \geq \frac{1}{2}$, then $\boldsymbol{P}(C_\infty) = 1$.*

7. Application to Random Power Series

Definition 7.1. Given a space Ω equipped with a σ-algebra $\mathcal{F}$ of Ω, we say a function $f : \Omega \to \mathbb{R}$ is *measurable* if for every Borel set $B \in \mathcal{B}$,

$$f^{-1}(B) \in \mathcal{F}. \tag{7.2}$$

Definition 7.3. A *random variable* on a probability triple $(\Omega, \mathcal{F}, \mathbf{P})$ is a function that is measurable on the probability space.

In other words, for every Borel set B, the event $\{\omega : X(\omega) \in B\} \in \mathcal{F}$ and hence has a well-defined probability.

Note that $\sigma(X)$, where X is a random variable, denotes the smallest σ-algebra of subsets of Ω such that X is measurable defined on this domain [8]. Then we have:

Theorem 7.4. *(Kolmogorov's Zero-One Law) Let $X_1, X_2, X_3, \ldots$ be a sequence of independent random variables. Then any event A such that*

$$A \in \bigcap_{i=1}^{\infty} \sigma(X_i, X_{i+1}, X_{i+2}, \ldots) \tag{7.5}$$

has probability 0 or 1.

Proof. The proof is nearly identical to that of the other formulation [9]. □

We consider randomly generated power series constructed via random variables.

Definition 7.6. Given a probability triple $(\Omega, \mathcal{F}, \mathbf{P})$ and a sequence of independent random variables $X_0, X_1, X_2, \ldots$, with each $X_i : \Omega \to \mathbb{R}$, then we define the *random power series*

$$f(z, w) = \sum_{i=0}^{\infty} X_i(\omega) z^i \tag{7.7}$$

for $z \in \mathbb{C}, \omega \in \Omega$ [10].

We will examine the convergence of random power series, and show that the radius of convergence is constant for all $\omega \in \Omega$ by Kolmogorov's Zero-One Law.

Definition 7.8. Given a sequence of real numbers $\{a_n\}$, define

$$\limsup a_n = \inf_{n \geq 0} \{\sup\{a_m : m \geq n\}\}. \tag{7.9}$$

We say this is the *limit superior* of the sequence [11].

It is a fact from analysis that for any complex power series of the form

$$f(z) = \sum_{i=0}^{\infty} a_n (z - c)^n, \tag{7.10}$$

with $a_i, c \in \mathbb{C}$, the radius of convergence of the series is

$$r = \frac{1}{\limsup \sqrt[n]{|a_n|}}. \tag{7.11}$$

Regarding our random power series, fix $\omega \in \Omega$. Then $f(z, \omega)$ is a complex power series as above with $c = 0$ [12]. If we define a new function

$$r(\omega) := \text{radius of convergence of } f(z, \omega), \tag{7.12}$$

it follows that

$$r(\omega) = \frac{1}{\limsup \sqrt[n]{|X_n(\omega)|}}. \tag{7.13}$$

Lemma 7.14. *For any sequence of random variables $X_0, X_1, X_2, \ldots$, $\limsup X_n$ is a random variable.*

Proof. It suffices to show that $\{\omega : \limsup X_n \leq x\} \in \mathcal{F}$ for all $x \in \mathbb{R}$. First note that for all $m \in \mathbb{N}$,

$$\{\omega : \sup_{n \geq m} X_n(\omega) \leq x\} = \bigcap_{n \geq m} \{\omega : X_n(\omega) \leq x\} \in \mathcal{F}, \tag{7.15}$$

where the last inclusion follows by the closure of σ-algebras under countable operations and the fact that each X_i is itself a random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Similarly we have

$$\{\omega : \inf_n X_n(\omega) \leq x\} = \bigcup_n \{\omega : X_n(\omega) \leq x\} \in \mathcal{F}. \tag{7.16}$$

It follows then that

$$\{\omega : \limsup X_n(\omega) \leq x\} = \bigcup_n \bigcap_{n \geq m} \{\omega : X_n(\omega) \leq x\} \in \mathcal{F} \tag{7.17}$$

and hence $\limsup X_n$ is a random variable defined on the probability space in its own right [13]. □

Now we show the radius of convergence is constant.

Theorem 7.18. *Given a sequence of independent random variables $X_1, X_2, \ldots$ with tail field τ, any random variable Y that is τ-measurable is constant with probability one* [14].

Proof. Given

$$\tau = \bigcap_{i=1}^{\infty} \sigma(X_i, X_{i+1}, X_{i+2}, \ldots), \tag{7.19}$$

and Y measurable with respect to τ, we have

$$\{\omega : Y(\omega) \in B\} \in \tau \tag{7.20}$$

for all Borel sets B. In considering intervals of the form $(-\infty, c]$, it follows that

$$\mathbf{P}(Y \leq c) \in \{0, 1\} \tag{7.21}$$

for all $c \in \mathbb{R}$. If Y is well defined (i.e. not $\pm\infty$ with probability one), then as we increase c there is some point where the probability defined above "jumps" from 0 to 1. Define this point as

$$x_0 := \inf \{c : \mathbf{P}(Y \leq c) = 1\} \tag{7.22}$$

Then we have that $\mathbf{P}(Y = x_0) = 1$ [14].

We will illustrate this precisely. Define

$$x_1 := \text{smallest } c \text{ such that } \mathbf{P}(Y > c) = 0. \tag{7.23}$$

We have three cases: $x_0 < x_1$, $x_0 > x_1$, $x_0 = x_1$. Suppose $x_0 < x_1$. Then there exists $\epsilon > 0$ such that

$$x_0 < x_1 - \epsilon < x_1. \tag{7.24}$$

Then we have $\mathbf{P}(Y > x_1 - \epsilon) = 1$, because it cannot be 0 by definition of x_1. Yet by definition of x_0, we must also have $P(Y \leq x_1 - \epsilon) = 1$. This is a clear contradiction as it implies

$$\mathbf{P}(\Omega) = \mathbf{P}(Y > x_1 - \epsilon) + P(Y \leq x_1 - \epsilon) = 1 + 1 = 2. \tag{7.25}$$

Now suppose $x_0 > x_1$. As $\mathbf{P}(Y > x_1) = 0$, by complementation we must have $\mathbf{P}(Y \leq x_1) = 1$, an immediate contradiction that x_0 was the infimum of such numbers. So $x_0 = x_1$. We immediately have $\mathbf{P}(Y < x_0) = 0$; now we have $\mathbf{P}(Y > x_0) = \mathbf{P}(Y > x_1) = 0$ by definition. We get:

$$\mathbf{P}(\Omega) = \mathbf{P}(Y < x_0) + \mathbf{P}(Y > x_0) + \mathbf{P}(Y = x_0) = 1. \tag{7.26}$$

As the first two terms equal 0, this implies $\mathbf{P}(Y = x_0) = 1$, i.e. Y is constant with probability one. The only case missing, where $Y = \pm\infty$ with probability one, is trivial. $\square$

As $\{X_n\}$ was an arbitrary sequence of random variables, we can simply define a new sequence of random variables as $\{\sqrt[n]{|X_n|}\}$. From this and the fact that lim sup is a τ-measurable random variable, it follows that our original formula for the radius of convergence

$$r(\omega) = \frac{1}{\limsup \sqrt[n]{|X_n(\omega)|}} \tag{7.27}$$

is constant with probability 1 for all $\omega \in \Omega$ [10].

Acknowledgments. I would like to express my sincerest gratitude to my mentor, Michael Neaton, for guiding me through the variety of topics and providing a clarity of understanding that much of the source material was lacking; I would like to extend this gratitude as well to Dr. Babai for so graciously volunteering his time to teach the apprentice REU program. Thanks also to Jon DeWitt for originally suggesting Kolmogorov's Zero-One Law as a topic to pursue.

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