TMA4275 Lifetime Analysis (Spring 2014)
Exercise 6

Problem 1 – The two-parameter exponential distribution

The two-parameter exponential distribution has density

$$f(t; \theta, \gamma) = \frac{1}{\theta} \exp\left\{-\frac{t-\gamma}{\theta}\right\} \text{ for } t \geq \gamma$$

Assume we have a right censored sample (y_i, δ_i), $i = 1, \ldots, n$ from this distribution.

a) Find the log-likelihood function $l(\theta, \gamma)$ for these data.

b) Let $(\hat{\theta}, \hat{\gamma})$ be the maximum likelihood estimators of (θ, γ). Verify that

$$\hat{\gamma} \leq y_{(1)}$$

where $y_{(1)}$ is the smallest observed time among $y_1, \ldots, y_n$.

Then find explicit expressions for $(\hat{\theta}, \hat{\gamma})$. Show in particular that we always have $\hat{\gamma} = y_{(1)}$

Remark: For the above to hold, it must be assumed that all censorings take place at or after time γ.

c) In the lectures we have considered a likelihood method for constructing confidence intervals for one of two parameters in a model. The method uses the following:

Let $\hat{\gamma}(\theta)$ be the MLE of γ when θ is given. Then

$$W(\theta) = 2(l(\hat{\theta}, \hat{\gamma}) - l(\theta, \hat{\gamma}(\theta)))$$

is approximately χ_1^2 when θ is the true parameter. (Note that $\tilde{l}(theta) = l(\theta, \hat{\gamma}(\theta))$ is the so-called profile log likelihood of θ).

Explain how this can be used to construct a confidence interval for θ. Do the calculations of the interval as far as you get.

d) Use MINITAB to estimate the parameters when the Pike cancer data (see page 91 of Slides 9-draft from lectures) are assumed to follow a two-parameter exponential distribution.

e) Reconsider the remark in question b) above. Can you think of cases where censorings also before time γ are possible? In such cases, how should the analysis in b) be modified?

Problem 2 – Censoring and truncation

$n = 10$ units with exponentially distributed life times and MTTF$= \theta$ are put on test. At time $c = 10$ the test is ended (type I censoring), and $r = 4$ units have failed by that time. The observed lifetimes are

$$0.9,\ 2.8,\ 5.9,\ 7.4$$

a) Write down the likelihood function and compute the MLE for θ. Which are the assumptions behind this approach?

b) Assume now that at the end of the experiment ($c = 10$) one does not know how many units were put on test, but only knows that the experiment has gone for 10 time units, with $r = 4$ failures at the times given.

How can you write down a likelihood for this case? (Hint: This is right truncation, see page 2 of Slides 11 from lectures).

Which are the assumptions behind this likelihood?

c) Maximize the likelihood in (b) to find the MLE under the conditions given there.

Problem 3 – Estimation and testing in the gamma distribution

Assume that the lifetime T is gamma distributed with shape parameter 2, so that

$$f(t; \lambda) = \lambda^2 t e^{-\lambda t}$$

for $t > 0$, where $\lambda > 0$ is the unknown parameter. We have n independent observations $t_1, \ldots, t_n$ of T (no censoring).

a) Find the MLE $\hat{\lambda}$ for λ. What are the properties of this estimator?

b) What is the estimate for λ when $n = 10$, $\sum t_i = 180$ (months)? Also find an estimate for the standard deviation of $\hat{\lambda}$ (i.e., standard error).

Remark: MINITAB does not include statistical inference for the gamma distribution, so you need to compute this by yourself.

c) Perform a test based on the log likelihood for the hypotheses

$$H_0 :\ \lambda = 0.25 \text{ versus } H_1 :\ \lambda \neq 0.25$$

What is the conclusion if the significance level is 5%?

d) Make a confidence interval for λ using the loglikelihood method (the "1.92 confidence interval").