Exercise

Consider a type of machine for which failures occur according to an HPP with basic ROCOF λ. Suppose that due to different environments, m machines of this kind will have ROCOF, respectively, $a_1\lambda, a_2\lambda, \ldots, a_m\lambda$, where the a_j are unobserved quantities modeled as independent variables from a gamma-distribution with expected value 1 and variance δ.

Derive the likelihood function for λ when the machines are, respectively, observed during given lengths of time, τ_j $(j = 1, \ldots, m)$.

Finally, show how you can estimate the values of the aj by a Bayes argument.

SOLUTION:

For a single machine we have
ROCOF $= a\lambda$.
Thus, with observation time τ,
the likelihood is (given a)

$$L(a) = \left(\prod_{i=1}^{N(\tau)} a\lambda\right) e^{-a\lambda\tau}$$

$$= a^{N(\tau)} \lambda^{N(\tau)} e^{-a\lambda\tau}$$

Now a has density

$$f(a;\delta) = \frac{a^{\frac{1}{\delta}-1} e^{-\frac{a}{\delta}}}{\Gamma(\frac{1}{\delta})\,\delta^{\frac{1}{\delta}}} \quad ; a>0$$

[~~a~~ gamma-distr. with expected value 1 and variance δ]

Thus the unconditional likelihood is

$$L = \int_0^\infty L(a) f(a;\delta)\, da$$

$$= \frac{\lambda^{N(\tau)}}{\Gamma(\frac{1}{\delta})\,\delta^{\frac{1}{\delta}}} \underbrace{\int_0^\infty a^{N(\tau)+\frac{1}{\delta}-1} e^{-(\lambda\tau+\frac{1}{\delta})a}\, da}$$

Would be a gamma density if it was multiplied by

$$\frac{(\lambda\tau+\frac{1}{\delta})^{N(\tau)+\frac{1}{\delta}-1}}{\Gamma(N(\tau)+\delta)}$$

Thus

$$L = \frac{\lambda^{N(\tau)}}{\Gamma\left(\frac{1}{\delta}\right)\delta^{\frac{1}{\delta}}} \cdot \frac{\Gamma(N(\tau)+\delta)}{\left(\lambda\tau+\frac{1}{\delta}\right)^{N(\tau)+\delta}}$$

Since we have m systems with this likelihood, the total likelihood would be

$$\prod_{j=1}^{m} L_j$$

where

$$L_j = \frac{\lambda^{N_j(\tau_j)}}{\Gamma\left(\frac{1}{\delta}\right)\delta^{\frac{1}{\delta}}} \frac{\Gamma(N_j(\tau_j)+\delta)}{\left(\lambda\tau_j+\frac{1}{\delta}\right)^{N_j(\tau)+\delta}}$$

Thus all we need from the data is $N_j(\tau_j)$; $j=1,\dots,m$

We can also "estimate" the value of a_j for each system.

Then we consider $\underline{a}$ as a parameter, with prior distribution

$$\pi(a) = \frac{a^{\frac{1}{\delta}-1} e^{-\frac{a}{\delta}}}{\Gamma\left(\frac{1}{\delta}\right)\delta^{\frac{1}{\delta}}} \quad \text{for } a>0$$

$$\sim \text{gamma}\left(\tfrac{1}{\delta}, \tfrac{1}{\delta}\right)$$

The data are the process of failures, with likelihood

$$L(\text{data}|a) = a^{N(\tau)} \lambda^{N(\tau)} e^{-a\lambda\tau}$$

The posterior distribution of a is hence

$$\pi(a|\text{data}) \propto \pi(a)\, L(\text{data}|a)$$

$$\sim \text{gamma}\left(\frac{1}{\delta} + N(\tau),\ \frac{1}{\delta} + \lambda\tau\right)$$

which has expected value

$$\hat{a} = \frac{\frac{1}{\delta} + N(\tau)}{\frac{1}{\delta} + \lambda\tau} = \frac{1+\delta N(\tau)}{1+\delta\lambda\tau}$$

<u>Thus:</u> we estimate a_j by

$$\hat{a}_j = \frac{1+\hat{\delta} N_j(\tau_j)}{1+\hat{\delta}\hat{\lambda}\tau_j}$$