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Covariates and Random Effects in a Gamma Process Model with Application to Degradation and Failure

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Abstract. The gamma process is a natural model for degradation processes in which deterioration is supposed to take place gradually over time in a sequence of tiny increments. When units or individuals are observed over time it is often apparent that they degrade at different rates, even though no differences in treatment or environment are present. Thus, in applying gamma-process models to such data, it is necessary to allow for such unexplained differences. In the present paper this is accomplished by constructing a tractable gamma-process model incorporating a random effect. The model is fitted to some data on crack growth and corresponding goodness-of-fit tests are carried out. Prediction calculations for failure times defined in terms of degradation level passages are developed and illustrated.

Keywords: degradation, frailty, gamma process, random effects, reliability, repeated measures, wear

1. Gamma Process Models

1.1. Basic Gamma Process

Models that describe the process of deterioration or degradation in units or systems are of interest in their own right, and are also key ingredients in processes that determine "failure" events. As such, they play a central role in efforts to improve the durability, reliability, and maintainability of systems. For some recent discussion of degradation models and their uses, see Singpurwalla (1995), Meeker and Escobar (1998), and Bagdonavicius and Nikulin (2001).

For certain types of degradation processes a model involving independent non-negative increments is appropriate. One such is the gamma process (e.g., Singpurwalla 1995), which can be thought of as arising from a compound Poisson process of gamma-distributed increments in which the Poisson rate tends to infinity while the sizes of the increments tend to zero in proportion. In most degradation problems there is substantial heterogeneity between the degradation paths of different individuals or units, beyond what can be accounted for by conditioning on explanatory variables. It is common to introduce unit-specific random effects to model such variability; for example, see Lu and Meeker (1993) for growth curve models with random coefficients. The purpose of this paper is to extend the gamma process model by incorporating random effects or covariates and to explore its use as a degradation model.

Gamma processes and extensions to them have been used as models for degradation or damage to materials (e.g., Cinlar, 1980; Singpurwalla, 1995), for the accumulation of flows into dams or storage devices (e.g., Moran, 1959), and other phenomena. They have also been used as priors in Bayesian nonparametric estimation (e.g., Kalbfleisch, 1978; Dykstra and Laud, 1981).

Notationally, we consider non-negative-valued processes $\{y(t), t \geq 0\}$. In the setting here, $y(t)$ represents the measured degradation for an individual unit at time t. A gamma process has the following properties:

(i) the increments $\Delta y(t) = y(t + \Delta t) - y(t)$ are independent;
(ii) $\Delta y(t)$ has a gamma distribution $Ga\{\xi, \eta(t + \Delta t) - \eta(t)\}$, where $\eta(.)$ is a given, monotone increasing function.

Thus, with the convention that $y(0) = 0$ and $\eta(0) = 0$, $y(t)$ has distribution $Ga\{\xi, \eta(t)\}$ with mean $\xi\eta(t)$, variance $\xi^2\eta(t)$, and probability density $g\{y; \xi, \eta(t)\}$, where $g(.)$ is defined by

$$g(y; \xi, \eta) = \xi^{-1}(y/\xi)^{\eta-1}e^{-y/\xi}/\Gamma(\eta). \tag{1.1}$$

Suppose that $y(t)$ is observed at times $t_0 < t_1 < \cdots < t_m$, yielding values $y_0, y_1, \ldots, y_m$. Define the y-increments $\Delta y_j = y_j - y_{j-1}$ for $j = 1, \ldots, m$. Then by (i) and (ii) their joint distribution has probability density function

$$\prod_{j=1}^{m} g(\Delta y_j; \xi, \Delta\eta_j), \tag{1.2}$$

where $\Delta\eta_j = \eta(t_j) - \eta(t_{j-1}) = \eta_j - \eta_{j-1}$ for $j = 1, \ldots, m$.

The independent increments property makes the subsequent mathematical treatment quite tractable. We remark that any increasing independent increments process must be a jump process (Ferguson and Klass, 1972), and realizations of the gamma processes considered here consist of a countably infinite number of jumps over a finite interval. The jumps occur at random times (in the Poisson sense) and the distribution of jump sizes is typically long-tailed, with relatively many more small jumps than large ones.

Figure 1 shows the sample paths for realizations of two gamma processes of the type fitted to data in Section 4. The processes $\{y(t), t \geq 0\}$ are of necessity generated discretely, and for Figure 1 we obtained $y(t_j)$ for $t_j = 0, 0.001, 0.002, \ldots, 0.12$ by using (1.2) with $y(0) = 0$, and $\eta(t)$ and ξ having values similar to their estimates in Section 4. The variation in jump sizes seen in gamma processes describes degradation such as crack propagation well, along with other phenomena in which deterioration tends to grow in frequent "bursts".

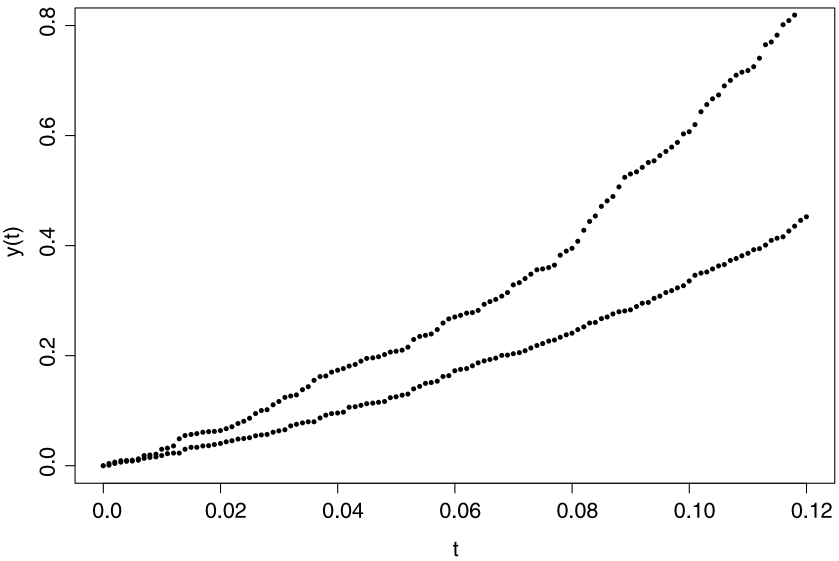

Figure 1. Two simulated gamma process sample paths.

1.2. Processes with Covariates and Random Effects

Covariates, represented by vector x, can be incorporated in the gamma process via an accelerated life model, replacing $\eta(t)$ by $\eta(te^{x^{\mathrm{T}}\beta})$, as in Bagdonavicius and Nikulin (2001). Alternatively, and more conveniently for the development below, x can be accommodated by taking $\xi = \xi(x)$; then x has the effect of rescaling $y(t)$ without changing the shape parameter of its gamma distribution. As noted above, random effects are also often needed to represent heterogeneity in the degradation paths of distinct units. One such model can be specified by replacing $\xi(x)$ by $z\xi(x)$, where z is a random effect. Mathematically tractable distributions result if we adopt a model in which $w = z^{-1}$ has distribution $Ga(\gamma^{-1}, \delta)$; w then has mean δ/γ and variance δ/γ^2, and z has finite mean $\mu_z = \gamma/(\delta - 1)$ for $\delta > 1$ and finite variance $\sigma_z^2 = \gamma^2/\{(\delta-1)^2(\delta-2)\}$ for $\delta > 2$. Then (1.1) gives the conditional density, say $f(y|z)$, and the marginal density of $y(t)$ follows as

$$f(y) = \int_0^\infty g(y; w^{-1}\xi_x, \eta_t) g(w; \gamma^{-1}, \delta) dw = B(\eta_t, \delta)^{-1} (\gamma\xi_x)^\delta y^{\eta_t - 1} / (y + \gamma\xi_x)^{\eta_t + \delta}, \tag{1.3}$$

in which we have written ξ_x for $\xi(x)$ and η_t for $\eta(t)$, and $B(\eta_t, \delta) = \Gamma(\eta_t) \times \Gamma(\delta)/\Gamma(\eta_t + \delta)$ is the beta function. Since (1.3) is a function of $\gamma\xi_x$ we henceforth write ξ_x without any constant term; this means that in the case with no covariates we take $\xi_x = 1$. An alternative parameterization that is often used in such situations is to let $\gamma = \delta$ and include the constant term γ in ξ_x. In this case w has mean 1 and variance δ^{-1}. Clearly there is no difference in the parameterization of (1.3) in this case.

We note that $\delta y(t)/(\gamma\xi_x\eta_t)$ has an F distribution, $\mathcal{F}_{2\eta_t,2\delta}$, and that $y(t)/\{\gamma\xi_x + y(t)\}$ has a beta distribution, $Be(\eta_t, \delta)$, which has density $B(\eta_t, \delta)^{-1}x^{\eta_t-1}(1-x)^{\delta-1}$, $0 < x < 1$. Further, moments of $y(t)$ can be evaluated as

$$\begin{aligned} E\{y(t)^r\} &= (\gamma\xi_x)^r\Gamma(\eta_t + r)\Gamma(\delta - r)/\{\Gamma(\eta_t)\Gamma(\delta)\} \\ &= \prod_{j=1}^{r}\{\gamma\xi_x(\eta_t + r - j)/(\delta - j)\} \quad \text{for } r < \delta; \end{aligned}$$

for $r \geq \delta$, $\mathrm{E}\{y(t)^r\} = \infty$. Thus, $y(t)$ has finite mean

$$E\{y(t)\} = \gamma\xi_x\eta_t/(\delta - 1) = \mu_z\xi_x\eta_t \quad \text{for } \delta > 1,$$

and finite variance

$$\begin{aligned} \mathrm{var}\{y(t)\} &= (\gamma\xi_x)^2\eta_t(\eta_t + \delta - 1)/\{(\delta-1)^2(\delta-2)\} \\ &= \sigma_z^2\xi_x^2\eta_t(\eta_t + \delta - 1) \quad \text{for } \delta > 2. \end{aligned}$$

Suppose now that $y(t)$ is recorded for a unit at times $t_0, t_1, \ldots, t_m$, yielding observations $y_0, \ldots, y_m$. The y-increments are defined as $\Delta y_j = y_j - y_{j-1}$ for $j = 1, \ldots, m$. Conditionally on their common random effect, z, they are independent as in (1.2), Δy_j now having distribution $Ga(z\xi_x, \Delta\eta_j)$, where $\Delta\eta_j = \eta_j - \eta_{j-1}$ with $\eta_j = \eta(t_j)$. Thus, as in the derivation of (1.3), the joint density of the Δy_j's is

$$\begin{aligned} f(\Delta y_1, \ldots, \Delta y_m) &= \int\left\{\prod_{j=1}^{m} g(\Delta y_j; w^{-1}\xi_x, \Delta\eta_j)\right\}g(w; \gamma^{-1}, \delta)dw \\ &= (\gamma\xi_x)^\delta\Gamma(\delta + \Delta\eta_+)\Bigg/\left\{\Gamma(\delta)\prod_{j=1}^{m}\Gamma(\Delta\eta_j)\right\} \\ &\quad \times\left\{\prod_{j=1}^{m}(\Delta y_j)^{\Delta\eta_j - 1}/(\gamma\xi_x + \Delta y_+)^{\delta + \Delta\eta_+}\right\}, \end{aligned} \tag{1.4}$$

where $\Delta y_+ = \sum_{j=1}^{m}\Delta y_j = y_m - y_0$ and $\Delta\eta_+ = \sum_{j=1}^{m}\Delta\eta_j = \eta_m - \eta_0$.

Note also for future reference that $\delta\Delta y_j/(\gamma\xi_x\Delta\eta_j)$ has distribution $\mathcal{F}_{2\Delta\eta_j,2\delta}$, and $r_j = \Delta y_j/(\gamma\xi_x + \Delta y_j)$ has distribution $Be(\Delta\eta_j, \delta)$. Moments of Δy_j are given by merely replacing η_t with $\Delta\eta_j$ in the expression for $E\{y(t)^r\}$ above.

The increments Δy_j for an individual are independent, given x and z, but, conditional only on x, they are correlated. The joint distribution (1.4) for $(\Delta y_1, \ldots, \Delta y_m)$ is a multivariate $\mathcal{F}$ distribution, and for $j \neq k$ and $\delta > 2$,

$$\mathrm{cov}(\Delta y_j, \Delta y_k) = (\gamma\xi_x)^2\Delta\eta_j\Delta\eta_k/\{(\delta-1)^2(\delta-2)\}.$$

The remainder of the paper is as follows. In Section 2 we give likelihood functions that can be used for estimation or testing of parameters in the gamma process models, and discuss inference procedures. Section 3 develops distributions for failure time variables T that are defined by the instant that the degradation $y(t)$ for a unit exeeds a threshold value y^F. A notable feature of the model considered here is that the predictive distribution for failure time, conditional on survival and observed degradation measurements up to some time t, are easily obtained. Section 4 fits the gamma model to some data on crack growth (Hudak et al., 1978). Section 5 presents a score test for assessing whether there is unit to unit heterogeneity, and Section 6 makes some concluding remarks.

2. Repeated Measures and Estimation

Parameters $\boldsymbol{\theta}$ involved in the specification of $y(t)$ in the models of Section 1 are γ, δ and any parameters involved in the specification of $\eta(t)$ and $\xi(x)$. Suppose that degradation processes $\{y_i(t), t \geq 0\}$ with associated covariates x_i are observed for n independent units $i = 1, \ldots, n$, with the i-th process observed at times $t_{0i} < t_{1i} < \cdots < t_{im_i}$. The joint density of the increments $\Delta y_{ij} = y_i(t_{ij}) - y_i(t_{i,j-1}), j = 1, \ldots, m_i$ is then given by (1.4) with $x = x_i, m = m_i, \Delta y_i = \Delta y_{ij}$, and $\Delta\eta_j = \Delta\eta_{ij} = \eta(t_{ij}) - \eta(t_{i,j-1})$. The product of such terms (1.4) across $i = 1, \ldots, n$ gives the likelihood function for the observed data, and inference about model parameters can be based on the usual maximum likelihood procedures. Note that since (1.4) only involves the differences $\Delta\eta_j$, the function $\eta(t)$ is identified only up to an arbitrary additive constant, i.e., $\alpha + \eta(t)$ gives the same likelihood as $\eta(t)$.

The likelihood based on terms (1.4) is still valid under an observation scheme where the final observation time t_{im_i} is not fixed, but depends on the process $y_i(t)$ having reached a certain level. For example, t_{im_i} could be the first planned observation time after $y_i(t)$ crosses a certain level. The reason that (1.4) is valid as a likelihood in this case is that the mechanism for ceasing observation depends only on the unit's previous degradation measurements, and so is a "stopping time" (see e.g., Andersen et al., 1993, pp. 61,81). More generally, the same likelihood is valid under observation schemes where the time of the next degradation measurement is decided on the basis of measurements already made.

The simplest approach to maximizing likelihoods $L(\boldsymbol{\theta})$ composed of terms (1.4) is to use a procedure not requiring (log) likelihood derivatives. There are many software packages that provide this, and also give variance estimates based on $(-\partial^2 \log L / \partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T)$ obtained by numerical differentiation.

For some purposes, such as 'tracking' individuals (e.g., Crowder and Hand, 1990, Section 9.3), we need estimates of the random effects, z, for individual units. One such estimate is $\tilde{z} = y_m / \xi_x \eta_m$, suggested by $\mathrm{E}(y_m|z) = z\xi_x\eta_m$. In fact, $\tilde{z}$ is the minimum-variance unbiased linear combination of the Δy_j ($j = 1, \ldots, m$): this follows from $\mathrm{var}(\Delta y_j|z) = (z\xi_x)^2 \Delta\eta_j$. An alternative estimator can be obtained from the joint distribution of $w = z^{-1}$ and $\Delta y_1, \ldots, \Delta y_m$. The corresponding density is $\{\prod_{j=1}^m g(\Delta y_j; w^{-1}\xi_x, \Delta\eta_j)\} g(w; \gamma^{-1}, \delta)$, from which the conditional distribution of w

given $(\Delta y_1, \ldots, \Delta y_m)$ follows as $Ga\{(y_m+\gamma)^{-1}, \xi_x \eta_m + \delta\}$, and then the conditional mean as

$$E(w \mid \Delta y_1, \ldots, \Delta y_m) = (\xi_x \eta_m + \delta)/(y_m + \gamma).$$

Parameter estimates may be inserted into this expression or $\tilde{z}$ above, to give an estimate of the realized (but unobserved) w or z for an individual unit.

We also note how the likelihood function can be obtained in the case where, instead of the degradation $y(t)$ being measured at specified times, the times $t_1 < \cdots < t_m$ at which $y(t)$ reaches specified levels $a_1 < \cdots < a_m$ are recorded. Let $\Delta t_j = t_j - t_{j-1}$ $(j = 1, \ldots, m)$ with t_0 given; then,

$$\begin{aligned} P(\Delta t_j \leq s \mid t_1, \ldots, t_{j-1}) =& P\{\Delta t_j \leq s \mid y(t_l) = a_l, \quad l = 1, \ldots, m-1\} \\ =& P\{\Delta y_j(s) \geq \Delta a_j \mid y_{j-1}\}, \end{aligned} \tag{2.1}$$

where $\Delta a_j = a_j - a_{j-1}$, $y_{j-1} = y(a_{j-1})$ and $\Delta y_j(s) = y(t_{j-1} + s) - y_{j-1}$. It is shown in Section 3 that, given y_{j-1}, the distribution of $\Delta y_j(s)$ can be represented as

$$\left(\frac{\eta_j + \delta}{\Delta \eta_j}\right)\left(\frac{\Delta y_j(s)}{y_{j-1} + \gamma \xi_x}\right) \sim F_{2\Delta\eta_j, 2\eta_{j-1}+2\delta}, \tag{2.2}$$

where $\eta_{j-1} = \eta(a_{j-1})$ and $\Delta \eta_j(s) = \eta_j(t_{j-1} + s) - \eta_{j-1}$. Thus, by (2.1) and (2.2),

$$P(\Delta t_j \leq s \mid t_1, \ldots, t_{j-1}) = 1 - F\left(\frac{(\eta_{j-1} + \delta)\Delta a_j}{\Delta \eta_j(s)(y_{j-1} + \gamma \xi_x)}\right), \tag{2.3}$$

where F is the distribution function of $F_{2\Delta\eta_j(s), 2\eta_{j-1}+2\delta}$. Using (2.1) and (2.3) we can obtain the density of Δt_j given $t_1, \ldots, t_{j-1}$, and then the likelihood function as

$$f(\Delta t_1, \ldots, \Delta t_m) = \prod_{j=1}^{m} f(\Delta t_j \mid t_1, \ldots, t_{j-1}). \tag{2.4}$$

The main difficulty in computing (2.1) is the inconvenience of differentiating (2.3) with respect to s, which occurs in both the argument and the first degree of freedom parameter of F. The simplest approach to maximum likelihood estimation by (2.4) is to use a maximisation procedure not requiring likelihood derivatives, and to compute the density $f(\Delta t_j \mid t_i, \ldots, t_{j-1})$ from (2.3) by numerical differentiation or equivalently, to replace the density by $F(\Delta t_j + \epsilon \mid t_1, \ldots, t_{j-1}) - F(\Delta t_j - \epsilon \mid t_1, \ldots, t_{j-1})$.

3. Distribution of Threshold-defined Failure Times

In many applications (e.g., Lu and Meeker, 1993; Whitmore and Schenkelberg, 1997) the failure time T for a unit is defined as the time at which its degradation path first crosses a threshold value y^F. In some models involving random effects this distribution is hard to obtain and it has then been found necessary to define the failure time in terms of the expected degradation path, which is not observable (e.g., Lu and Meeker, 1993). For the gamma process model considered here the sample paths are monotonic and the failure-time distribution is easily obtained. In particular, the results of Section 1 give

$$P(T > t) = P\{y(t) < y^F\} = F\{\delta y^F/(\gamma\xi_x\eta_t)\}, \tag{3.1}$$

where F is the distribution function of $\mathcal{F}_{2\eta_t,2\delta}$.

One of the advantages of measured degradation is its use in predicting time to failure. Suppose that a unit is unfailed at time t and that we are interested in whether it will survive beyond time $t + \Delta t$, for some $\Delta t > 0$. If we do not know the current degradation level, $y(t)$, then we can use

$$P(T > t + \Delta t \mid T > t) = P(T > t + \Delta t)/P(T > t), \tag{3.2}$$

which is easily obtained from (3.1), or from any other model for the distribution of T. If we do know $y(t)$, however, conditioning on it would be expected to give more precise predictions than (3.2): it is readily seen that

$$P\{T > t + \Delta t \mid T > t, y(t)\} = P\{\Delta y(t) < y^F - y(t) \mid y(t)\}, \tag{3.3}$$

where $\Delta y(t) = y(t + \Delta t) - y(t)$. From (1.4) it follows that the conditional density of $\Delta y(t)$ given $y(t)$ is

$$f\{\Delta y(t) \mid y(t)\} = B(\eta_t + \delta, \Delta\eta_t)^{-1} \Delta y(t)^{\Delta\eta_t - 1} \times \{y(t) + \gamma\xi_x\}^{\eta_t+\delta}/\{y(t) + \Delta y(t) + \gamma\xi_x\}^{\eta_t+\Delta\eta_t+\delta},$$

where $\Delta\eta_t = \eta_{t+\Delta t} - \eta_t$. That is, conditional on $y(t)$,

$$\left(\frac{\eta_t + \delta}{\Delta\eta_t}\right)\left(\frac{\Delta y(t)}{y(t) + \gamma\xi_x}\right) \sim \mathcal{F}_{2\Delta\eta_t,2\eta_t+2\delta} \tag{3.4}$$

and, thus, by (3.3),

$$P\{T > t + \Delta t \mid T > t, y(t)\} = F\left(\frac{(\eta_t + \delta)\{y^F - y(t)\}}{\Delta\eta_t\{y(t) + \gamma\xi_x\}}\right), \tag{3.5}$$

where F is the distribution function of $\mathcal{F}_{2\Delta\eta_t,2\eta_t+2\delta}$. Note that (3.5) reduces to (3.1) in the case where $t = 0, y(0) = \eta_0 = 0$ and we associate Δt with t in (3.1).

Estimates of failure time distributions are obtained by inserting estimates of γ, δ and parameters in η_t and ξ_x in the expressions (3.1) and (3.5). Interval estimation by the usual asymptotic normal likelihood methods is messy because of the complexity of derivatives of (3.1) and (3.5) with respect to the parameters. A more attractive alternative is to use parametric bootstrap methods (see Davison and Hinkley, 1997, chapters 1 and 5).

4. Hudak Crack-growth Data

Hudak et al. (1978) presented some crack-growth data, which is listed in Lu and Meeker (1993) and illustrated here in Figure 2. There are 21 metallic specimens, each subjected to loading cycles, with the crack lengths recorded every 10^4 cycles. Initial crack lengths were 0.9 inches for each specimen, and Lu and Meeker considered

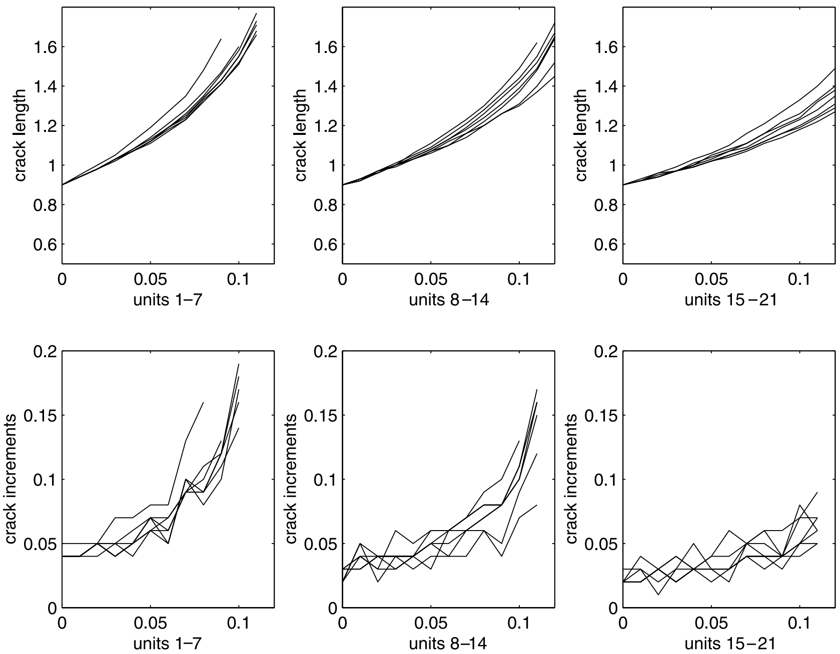

Figure 2. Crack growth data from Hudak et al. (1978): 21 units, up to 13 crack-length measurements on each.

failure to occur when crack length exceeded $y^F = 1.6$ inches. The data give crack length measurements $y(t)$ up to 120,000 cycles or the first measurement after failure, whichever occurred first. For the discussion here we take $t =$ no.cycles $/10^6$, so the potential degradation measurement times are $t_j = j/100$ for $j = 0, 1, \ldots, 12$.

The data were used by Lu and Meeker (1993) and Robinson and Crowder (2000) to illustrate statistical methodology. Both papers fitted a Paris–Erdogan curve to the repeated measures: for this model the crack length at time t is given as $y_0(1 - y_0^{\beta_2}\beta_1\beta_2 t)^{-\beta_2^{-1}}$, where y_0 is the initial crack length, here 0.9 in., and the βs are parameters that vary randomly across units. We use the data again to illustrate the present methodology. There are no covariates here and so in accordance with the previous model we take $\xi_x = 1$ and

$$\eta(t) = \beta_0(1 - y_0^{\beta_2}\beta_1\beta_2 t)^{-\beta_2^{-1}}. \tag{4.1}$$

Since a stopping rule was applied to the data whereby recordings ceased if the crack length exceeded 1.6, thus, m varies between specimens. The likelihood construction based on (1.4) is still valid in this case, as discussed in Section 2. Note that the likelihood depends only on the values of $y(t) - y(0) = y(t) - 0.9$ and of $\eta(t) - \eta(0)$, for $t = 0.01, 0.02, \ldots$ On fitting the model (4.1) it was found that the parameter β_2 tended towards zero during iterative maximisation of the log-likelihood. The limit of (4.1) as $\beta_2 \to 0$ is $\beta_0 e^{\beta_1 t}$: the modified model

$$\eta(t) = e^{\beta_0 + 10\beta_1 t} \tag{4.2}$$

was then fitted, in which the parameter β_0 has been moved into the exponent and the factor 10 has been inserted, both for convenience of scaling. Maximum likelihood estimates were obtained as follows (with standard errors in parentheses):

$$\hat{\beta}_0 = 4.48(0.11), \qquad \hat{\beta}_1 = 1.20(0.04),$$
$$\log \hat{\gamma} = -3.95(0.55), \qquad \log \hat{\delta} = 2.14(0.50).$$

In Figure 3 the first column shows, for an illustrative set of units ($i = 3, 8, 17, 20$), the observed values, y_{ij}, together with the fitted curves, vs. t_j. The latter were computed from $\eta_i(t) = z_i \eta(t)$, giving fitted values

$$\tilde{y}_{ij} = \mathrm{E}(y_{ij}|z_i, x) = z_i \eta_j, \quad i = 1, \ldots, 21, \;\; j = 1, \ldots, m_i,$$

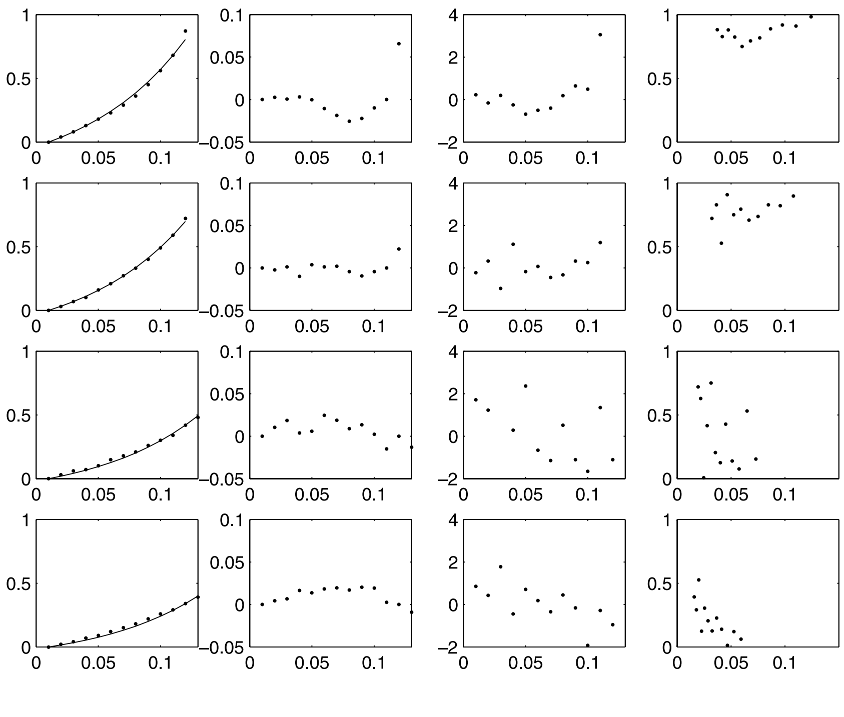

Figure 3. Fitted curves and residuals for units 3, 8, 17, 20. Column 1: observed crack lengths plus fitted curves vs. t; column 2: fitted residuals vs. t; column 3: standardised increment residuals vs. t; column 4: uniform increment residuals vs. estimated increments.

in which z_i is replaced by an estimate described in Section 2 and the parameters are replaced by their maximum likelihood estimates. In the second column the corresponding residuals, $y_{ij} - \tilde{y}_{ij}$, are plotted vs. t_j. Column 3 gives plots of the standardised increment residuals vs. t_j; these are defined as $(\Delta y_{ij} - \Delta\tilde{y}_{ij})/sd_{ij}$, where sd_{ij} is an estimate of $z_i\sqrt{\Delta\eta_j}$. In the fourth column uniform increment residuals are plotted against estimated Δy_{ij}-values. These uniform residuals are defined via the probability integral transform: $r_{ij} = \Delta y_{ij}/(\gamma + \Delta y_{ij})$ has a beta distribution, $Be(\Delta\eta_{ij}, \delta)$, and so

$$u_{ij} = B(r_{ij}; \Delta\eta_{ij}, \delta)/B(1; \Delta\eta_{ij}, \delta)$$

has a standard uniform distribution, $U(0,1)$, where $B(y; \eta, \delta) = \int_0^y t^{\eta-1}(1-t)^{\delta-1}dt$ is the incomplete beta function.

The gamma model here represents between-unit variation quite well, and the small but systematic lack of fit seen in Figure 3 could be addressed by allowing the parameter β_1 in (3.2) to vary across units. This is essentially what is done in the growth curve model of Lu and Meeker (1993). We leave an investigation of this aspect for future work.

The failure-time distribution function, $F_T(t)$, is easily estimated by inserting parameter estimates in (3.1). The maximum likelihood estimate of $F_T(t)$ is plotted against t in Figure 4, along with the Kaplan–Meier estimate, computed by Lu and Meeker from extended follow-up of each unit until it crossed the failure threshold.

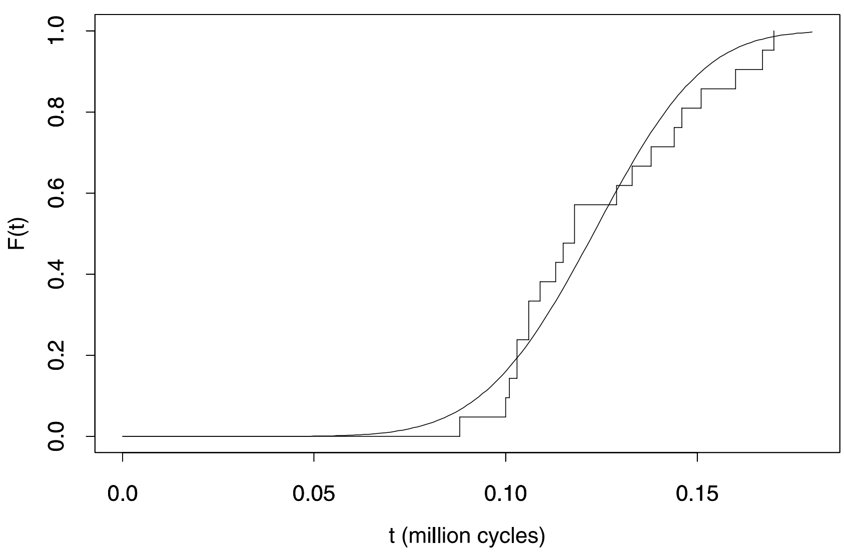

Figure 4. Estimates of the failure time distribution function $F_T(t)$.

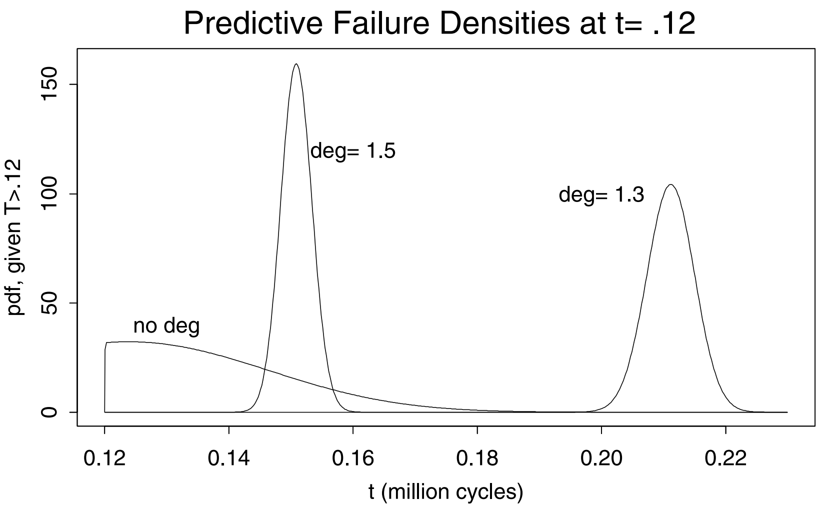

Figure 5. Predictive failure-time densities.

The estimated curve, based on the gamma-process model, appears to follow the data reasonably well, and is close to an estimate obtained by Lu and Meeker (1993) for their random effects growth curve model.

Figure 5 shows the effect on prediction of conditioning on the current degradation level. Three density curves are plotted: the first, labelled 'no deg', is $f_T(t \mid T > 0.12)$, the second is $f_T\{t \mid T > 0.12, y(0.12) = 1.5\}$, and the third is $f_T\{t \mid T > 0.12, y(0.12) = 1.3\}$. Not surprisingly, the effect of conditioning on the current degradation level, $y(0.12)$ is striking.

Virkler et al. (1979) gave similar crack-growth data on 68 specimens, with 164 observations on each. They give the times at which crack size reached a series of levels, $a_1, \ldots, a_m$, so one should in this case use the likelihood function given by (2.4) and (2.5). The gamma-process model provides a fit to these data that is qualitatively very similar to the fit for the Hudak data.

5. A Score Test for the Presence of Random Effects

An extreme case of the model, of importance in practice, occurs when there is no random variation between individual units in their stochastic behaviour. Thus, we consider the null hypothesis $H_0: \sigma_w^2 = 0$, i.e., the variance of the random effects distribution for $w = z^{-1}$ is zero. The situation can be realised by letting $\gamma \to \infty$ with $\delta/\gamma = E(w) = v$ fixed: then the w-distribution degenerates at the single value v and the model reverts back to the basic gamma process, with the Δy_j having joint density obtained from (1.1) as

$$f_0(\Delta y_1, \ldots, \Delta y_m) = \prod_{j=1}^{m} g(\Delta y_j; \xi_x/v, \Delta\eta_j) = \prod_{j=1}^{m} \{\Gamma(\Delta\eta_j)^{-1}(v/\xi_x)^{\Delta\eta_j} \Delta y_j^{\Delta\eta_j - 1} e^{-\Delta y_j v/\xi_x}\}. \tag{5.1}$$

To test for this situation we will derive a score statistic as follows. Let $l = l(\beta, \gamma, \delta)$ be the logarithm of the likelihood function for a single individual, as given by (2.1) with β denoting the parameter vector governing $\eta(.)$, and replace δ by γv. Then

$$\begin{aligned} \partial l/\partial\gamma = {} & v\{1 + \log(\gamma\xi_x) + \psi(\gamma v + \eta_m) - \psi(\gamma v) - \log(\gamma\xi_x + y_m)\} \\ & - \xi_x(\gamma v + \eta_m)(\gamma\xi_x + y_m)^{-1}, \end{aligned} \tag{5.2}$$

in which $\Delta\eta_+$ has been replaced by η_m (taking $\eta_0 = 0$) and Δy_+ has been replaced by y_m (taking $y_0 = 0$), and where $\psi(.)$ is the digamma function. As $\gamma \to \infty$, $\partial l/\partial\gamma \to 0$, so a more detailed calculation is called for. Expanding about $\gamma = \infty$ gives $\partial l/\partial\gamma = A\gamma^{-2} + \mathrm{O}(\gamma^{-3})$, where

$$A = -\frac{1}{2} v\xi_x^{-2}\{(y_m - \eta_m\xi_x/v)^2 - \eta_m\xi_x^2/v^2\}. \tag{5.3}$$

The asymptotic form $\psi(\gamma) \sim \log\gamma - \frac{1}{2\gamma} - \frac{1}{12\gamma^2}$ has been used here (e.g., Abramowitz and Stegun, 1965, eqn. 6.3.18). Under the full distribution, given in (2.1), the mean of the leading term can be derived as

$$\mathrm{E}(A) = -\frac{1}{2} v\eta_m[\gamma^2(\eta_m + \delta - 1)/\{(\delta-1)^2(\delta-2)\} + \eta_m/\{v(\gamma v - 1)\}^2 - 1/v^2]. \tag{5.4}$$

This tends to zero as $\gamma \to \infty$ with $\delta/\gamma = v$. Under the null distribution, given by (5.1), the mean $\mathrm{E}_0(A) = 0$ and the variance is

$$\begin{aligned} \mathrm{var}_0(A) = {} & (2v\xi_x^2)^{-2}[\eta_m^{(4)} - 4\eta_m^{(3)}(\xi_x\eta_m) + 6\eta_m^{(2)}(\xi_x\eta_m)^2 \\ & - 4\eta_m^{(1)}(\xi_x\eta_m)^3 + (\xi_x\eta_m)^4 - (\xi_x^2\eta_m)^2], \end{aligned} \tag{5.5}$$

where we have used the notation $\eta_m^{(r)} = \eta_m(\eta_m + 1)\ldots(\eta_m + r - 1)$ for $r = 1, 2, 3, 4$.

Application of this to the Hudak data yields the following results. The parameter estimates (with their standard errors) for the null model, (5.1), are $\hat{\beta}_0 = 3.53$ (0.14), $\hat{\beta}_1 = 1.03$ (0.07) and $\log\hat{v} = 4.78$ (0.92), and the normal deviate value is $\hat{A}/se(A) = -20.53$, where $se(A)$ is estimated from $\mathrm{var}_0(A)$; thus, there is strong evidence for rejecting the null hypothesis of zero variation in the random effects, as suggested by the degradation curves in Figure 2.

Whitmore and Schenkelberg, (1997) fitted a Wiener process model to some data on the degradation of heating cables. The measure of degradation is electrical resistance, which rises with age. This process may have smoother sample paths than seen with gamma processes. However, for illustration we consider the test of homogeneity above for these data also. Following the model fitted in the paper, we take $\xi_x = 1$

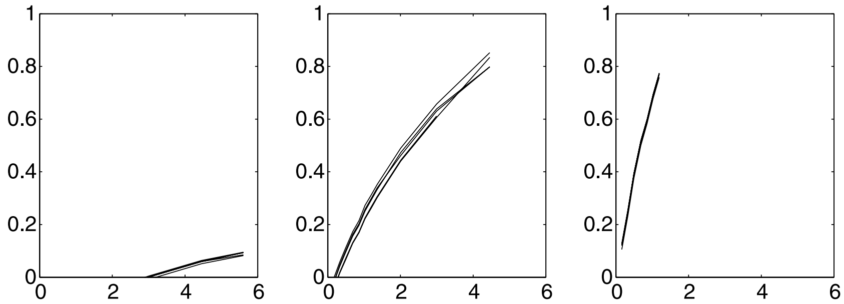

Figure 6. Heating cables data from Whitmore and Schenkelberg (1997): 5 specimens, 10 resistance measurements on each.

and $\eta(t) = e^{\beta_0}(1 - e^{-\beta_1 t})$, where t is the age of the cable in hours/1000. Figure 6 shows the degradation curves for three groups, each of five specimen cables, treated under different conditions: log-resistance is plotted here against t. The curves within groups are very close to each other, indicating little variation between individual processes.

The score test, applied to the three groups separately, produces values $\hat{A}/se(\hat{A})$ equal to 1.43, 0.35 and 1.57. Thus, there is not strong evidence against the null hypothesis of zero variance of the random effects.

6. Concluding Remarks

The gamma process developed in this paper has a number of convenient features as a degradation model. Individual processes are monotonic with conditionally independent increments, and the random effects incorporate heterogeneity across units. The probability distribution of sample path measurements at discrete time points has a simple closed form, as does the distribution of times to failure, defined by the degradation measure's crossing a threshold level. In these respects it is similar to Wiener process models (e.g., Doksum and Hoyland, 1992, Whitmore, 1995) with random drift parameters (e.g., Whitmore, 1986; Aalen and Gjessing, 2001), which are convenient for settings where a marker or degradation process is not strictly monotone.

We have not considered any specific regression models for $\xi(x)$ in this paper, nor situations where covariates are involved. We hope to do this in a future article, and note that one important area of application is to accelerated degradation testing (e.g., Meeker and Escobar, 1998, chapter 21), where covariates represent stresses that accelerate degradation. An obvious regression function specification is $\xi(\mathbf{x}) = \exp(\boldsymbol{\beta}'\mathbf{x})$, where $\mathbf{x}$ and $\boldsymbol{\beta}$ are vectors of covariates and regression coefficients, respectively.

The model here can be extended to allow covariates or random effects in the $\eta(t)$ component, as indicated in Section 1. Dealing with random effects in $\eta(t)$ is computationally more difficult and we defer a discussion of this topic to a subsequent article. We note that the analyses of Sections 4 and 5 suggest that a slight degree of unit-to-unit variability in $\eta(t)$ might deal with the small but systematic departures from the current model that are seen in the examples.

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References

O. Aalen and H. K. Gjessing, "Understanding the shape of the hazard rate: a process point of view (with discussion)", *Statistical Science* vol. 16 pp. 1–22, 2001.

M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover: New York, 1965.

P. K. Andersen, O. Borgan, R. D. Gill and N. Keiding, *Statistical Models Based on Counting Processes*, Springer-Verlag: New York, 2001.

V. Bagdonavicius and M. Nikulin, "Estimation in degradation models with explanatory variables," *Lifetime Data Analysis* vol. 7 pp. 85–103, 2001.

E. Cinlar, "On a generalization of gamma processes," *Journal of Applied Probability* vol. 17 pp. 467–480, 1980.

M. J. Crowder and D. J. Hand, *Analysis of Repeated Measures*, Chapman and Hall: London, 1990.

A. C. Davison and D. V. Hinkley, *Bootstrap Methods and Their Application*, Cambridge University Press, Cambridge: U.K., 1997.

K. A. Doksum, and A. Hoyland, "Models for variable-stress accelerated life test experiments based on Wiener processes and the inverse Gaussian distribution," *Technometrics* vol. 34 pp. 74–82, 1992.

R. L. Dykstra and P. Laud, "A Bayesian nonparametric approach to reliability," *Annals of Statistics* vol. 9 pp. 356–367, 1981.

T. S. Ferguson and M. J. Klass, "A representation of independent increments processes without Gaussian components," *Annals of Mathematical Statistics* vol. 43 pp. 1634–1643, 1972.

S. J. Hudak, A. Saxena, R. J. Bucci, and R. C. Malcolm, "Development of standard methods of testing and analyzing fatigue crack growth rate data," *Technical Report AFML-TR-78-40*, Westinghouse R & D Center, Westinghouse Electric Corporation, Pittsburgh, PA, 1978.

J. D. Kalbfleisch, "Bayesian nonparametric analysis of survival data," *Journal of the Royal Statistical Society B* vol. 40 pp. 214–221, 1978.

C. J. Lu and W. Q. Meeker "Using degradation measurements to estimate a time-to-failure distribution," *Technometrics* vol. 35 pp. 161–174, 1993.

W. Q. Meeker and L. E. Escobar, *Statistical Methods for Reliability Data*, John Wiley and Sons: New York, 1998.

P. A. P. Moran, *The Theory of Storage*, Methuen: London, 1959.

M. E. Robinson and M. J. Crowder "Bayesian methods for a growth-curve degradation model with repeated measures," *Lifetime Data Analysis* vol. 6 pp. 357–374, 2000.

N. D. Singpurwalla, "Survival in dynamic environments," *Statistical Science* vol. 1 pp. 86–103, 1995.

D. A. Virkler, B. M. Hilberry, and P. K. Goel "The statistical nature of fatigue crack propagation," *Trans. ASME* pp. 148–153, 1979.

G. A. Whitmore, "Normal-gamma mixtures of inverse Gaussian distributions," *Scandanavian Journal of Statistics* vol. 13 pp. 211–220, 1986.

G. A. Whitmore, "Estimating degradation by a Wiener diffusion process," *Lifetime Data Analysis* vol. 1 pp. 307–319, 1995.

G. A. Whitmore and F. Schenkelberg, "Modelling accelerated degradation data using Wiener diffusion with a time scale transformation," *Lifetime Data Analysis* vol. 3 pp. 27–45, 1997.