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Competing Risks

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Abstract

Consider a unit which can experience any one of k competing failure types, and suppose that for each unit we observe the time to failure, T, and the type of failure, $C \in \{1, 2, \ldots, k\}$. The case of observing the pair (T, C) is termed "competing risks" in the statistical literature. After considering some examples we review basic notation and theory of competing risks. In particular we consider the latent failure time approach to competing risks in which the k risks are represented by potential failure times $T_1, \ldots, T_k$ where only the smallest, $T = \min_j T_j$, is observed together with its index $C = \arg\min_j T_j$. In reliability studies, the marginal distributions of the T_j are often of primary interest, but are unfortunately non-identifiable in general. Additional, though non-testable, assumptions to obtain identifiability are considered, as are bounds for the marginal distributions given in terms of observable functions. The likelihood function of right censored competing risks data is given and its consequences for both parametric and non-parametric estimation are explained. Extensions of the classical theory of competing risks to more general Markov models and to repairable systems are briefly discussed.

Keywords: Competing risks; Latent failure times; Sub-distribution function; Cause-specific hazard function; Identifiability; Preventive maintenance; Repairable system.

1 INTRODUCTION

Suppose that units under study can experience any one of several distinct failure types, and that for each unit we observe both the time to failure and the type of failure. Failure may here, for example, correspond to breakdown of a mechanical component where there are several possible root causes for the failure, such as vibration, corrosion, etc. While this is a typical case of a "competing risks" situation in reliability, the theory of competing risks does not originate from reliability theory. In fact, it can be traced back to David Bernoulli's attempts in 1760 to disentangle the risk of dying from smallpox from other causes. This is indeed a classical example of competing risks, where individuals are subject to multiple causes of death. Similar applications occur in demography and actuarial science, usually under the name of multiple-decrement analysis.

1.1 Formal definition of competing risks

Formally one observes the pair (T, C) where $T > 0$ is the time of failure and $C \in \{1, 2, \ldots, k\}$ represents the type of failure. It is thus assumed that there are k different failure types, and that each failure can be classified as belonging to exactly one of the k types. Note that "failure" is used as a generic term and may in practice correspond to any event of interest depending on the application at hand. Also, "time" need not mean calendar time, but can in principle be any suitable measurement which is non-decreasing with calendar time, such as operation time, number of cycles, number of kilometers run, length of a crack etc.

An intuitive way of describing a competing risks situation with k risks, is to assume that to each risk is associated a failure time T_j, $j = 1, ..., k$. These k times are thought of as the hypothetical failure times if the other risks were not present, and they are referred to as latent failure times. When all the risks are present, the observed time to failure of the system is the smallest of these failure times along with the actual cause of failure. Thus by letting $T = \min\{T_1, ..., T_k\}$ and $C = c$ if $T = T_c$, we observe the pair (T, C) and we are back to the formulation of the previous paragraph. Note that it is assumed that the T_c for which minimum is attained is uniquely given.

1.2 Uses of competing risks in reliability and maintenance studies

Crowder [11, Ch. 1] gives some simple examples of the uses of competing risks in reliability studies. One of these is taken from King [18] who studied data of breaking strengths of certain wire connections. Two types of failure were defined (so $k = 2$): breakage at the bonded end and breakage along the wire itself.

Modern reliability data banks usually distinguish between a large number of failure modes, which suggests the use of methods from the theory of competing risks. Cooke [8, 9] review some main styles in the design of **reliability databases**, as well as models and methods for their analysis.

Traditionally, competing risks were analyzed as if they were independent of each other. This assumption appears to be dubious in many reliability applications, however. Even if assumptions of stochastic independence may often be justified by the physically independent functioning of components, a dependence between risks may be introduced by many sources, for example **load sharing** between components, shared working environment, manufacture and maintenance.

A simple case of dependent risks occur in the case when a potential component failure at some time T_1 may be avoided by a preventive maintenance (PM) at time T_2 (see [9]). The assumption that T_1 and T_2 are independent is clearly unreasonable in this application, since the maintenance crew is likely to have some information regarding the component's state during operation, and this insight is used to perform maintenance with the aim of avoiding component failures. Note that the observable result is the pair (T, C), rather than the latent times T_1 and T_2 themselves, which are usually the times of primary interest. For example, knowing the distribution of T_1, the true failure time distribution, could be the basis for **maintenance optimization**. However, as will be discussed later, the marginal distributions of the T_j are not identifiable from observation of (T, C) alone, unless specific assumptions are made on the dependence.

1.3 Basic literature on competing risks

The recent book by Crowder [11] gives a comprehensive review of the the theory and methods of competing risks. An older book devoted to the subject is David and Moeschberger [12]. Several standard books on reliability and survival analysis contain chapters on competing risks, for example Lawless [19], Kalbfleisch and Prentice [17], Nelson [24], Bedford and Cooke [4] and Andersen et al. [2]. Among several review papers written on the subject we mention Gail [14] and Moeschberger and Klein [23].

2 MODEL SPECIFICATION

The joint distribution of the pair (T, C) from an individual is completely specified by the sub-distribution functions

$$F_j(t) = P(T \leq t, C = j); \quad t > 0,\ j = 1, ...k.$$

The corresponding sub-density functions, when they exist, are given by differentiation, $f_j(t) = F_j'(t)$. In the formulas given in the following, the ranges of t and j are clear and will be mostly suppressed.

The marginal distribution of T is given by the **cumulative distribution function**

$$F(t) = P(T \leq t) = \sum_{j=1}^{k} F_j(t),$$

or by the survival function $\bar{F}(t) = 1 - F(t)$. Note that here and in the sequel, a bar above a capital letter means that this is the (sub-)survival function corresponding to the (sub-)distribution function given without a bar. The marginal distribution of C is given by $\pi_j = P(C = j) = F_j(\infty)$.

The distribution of (T, C) can alternatively be specified by the sub-hazard functions, which when they exist are given by

$$\lambda_j(t) = \lim_{\Delta t \to 0} \frac{P\left(T \leq t + \Delta t, C = j | T > t\right)}{\Delta t} = \frac{f_j(t)}{\bar{F}(t)}. \tag{1}$$

It follows that $\lambda(t) = \sum_{j=1}^{k} \lambda_j(t)$ is the hazard function of T. Moreover, from equation (1) follows that $f_j(t) = \lambda_j(t)\bar{F}(t)$ which by integration gives the useful connection

$$F_j(t) = \int_0^t \lambda_j(u)\bar{F}(u)du. \tag{2}$$

Next, defining the cumulative sub-hazard functions as $\Lambda_j(t) = \int_0^t \lambda_j(u)du$ it is seen that $\Lambda(t) = \sum_{j=1}^k \Lambda_j(t)$ is the cumulative hazard function of T. Thus we have

$$\bar{F}(t) = e^{-\Lambda(t)} = e^{-\sum_{j=1}^k \Lambda_j(t)} = \prod_{j=1}^k \bar{G}_j^*(t) \tag{3}$$

where

$$\bar{G}_j^*(t) = e^{-\Lambda_j(t)}. \tag{4}$$

Note that $\bar{G}_j^*(t)$ is a survival function (possibly with an atom at infinity), but that it is not in general the distribution of any observable random variable (see, however, Section 3.1).

The sub-hazard functions $\lambda_j(t)$ have the intuitive interpretation as the failure rate from a specific cause conditional on survival up to time t. They are also known under the names mode-specific or cause-specific hazard functions, and have in older literature been called crude hazard rates.

2.1 Latent failure time representation

Consider again the representation where $T = \min\{T_1, ..., T_k\}$ and $C = c$ if $T = T_c$, with c assumed uniquely given. Let the joint survival function of $T_1, ..., T_k$ be $\bar{K}(t_1, \ldots, t_k) = P(T_1 > t_1, \ldots, T_k > t_k)$. Then the survival function of T can be evaluated as $\bar{F}(t) = \bar{K}(t, t, \ldots, t)$. The sub-density functions can also be calculated directly from the joint survival function as

$$f_j(t) = -\left(\frac{\partial \bar{K}(t_1, \ldots, t_k)}{\partial t_j}\right)_{t_1 = \ldots = t_k = t},$$

and it further follows that

$$\lambda_j(t) = \frac{f_j(t)}{\bar{F}(t)} = -\left(\frac{\partial \log \bar{K}(t_1, \ldots, t_k)}{\partial t_j}\right)_{t_1 = \ldots = t_k = t}. \tag{5}$$

Let the marginal survival function of T_j be denoted $\bar{G}_j(t) = P(T_j > t)$ and let the corresponding hazard rate function be $h_j(t) = -\bar{G}_j'(t)/\bar{G}_j(t)$. It is noted that in general $h_j(t)$ and $\lambda_j(t)$ are different and have different interpretations. While the former has traditionally been called the net rate, the latter is the crude rate as mentioned before. It will be seen later, however, that $h_j(t) = \lambda_j(t)$ for all $t > 0$ when the T_j are independent.

2.2 The identifiability problem

As already mentioned, the main interest in a competing risks analysis is often in the joint and marginal distributions of the latent failure times $T_1, ..., T_k$. The classical problem of competing risks is, however, that the distribution of the observable pair (T, C) does not in general determine the distribution of the latent failure times. This means that there are several different joint distributions of $T_1, ..., T_k$ which give rise to the same distribution of (T, C). This fact, called non-identifiability of the distributions of $T_1, \ldots, T_k$, was noted by Cox [10] for the case of two failure causes, while Tsiatis [27] studied the general case.

The main result of Tsiatis [27] states that if the set of sub-distribution functions $F_j(t)$ is given for some model with dependent risks, then there exists a unique model with independent risks yielding the same $F_j(t)$. This model is defined by the joint survival function

$$\bar{K}(t_1, \ldots, t_n) = \prod_{j=1}^{k} \bar{G}_j^*(t_j) \tag{6}$$

where the $\bar{G}_j^*(t)$ are given by equation (4). Thus, one cannot know, from observations of (T, C) alone, which of the two models is correct, since they will both fit the data equally well.

3 HOW TO DEAL WITH THE IDENTIFIABILITY PROBLEM

In this section we consider ways of overcoming the identifiability problem under the latent variable representation of competing risks. It should be stressed that this can only be done by imposing additional assumptions in the model. These may be of various kinds, but one should always have in mind that under observation of the pair (T, C), the assumptions will always be non-testable.

3.1 Assuming independent risks

The classical assumption is that the risks act independently, so that the latent failure times T_j are independent. It then follows from Tsiatis [27] (see above) that we have identifiability of the distributions in question (under regularity assumptions), meaning that the marginal distributions of the T_j now can be computed from the sub-distribution functions $F_j(t)$. In practice this means that the marginal distributions can be estimated in a consistent manner from competing risks data. Furthermore, in the case of independence we can write $\bar{K}(t_1, \ldots, t_k) = \prod_{j=1}^{k} \bar{G}_j(t_j)$ and hence it follows from equation (6) that $\bar{G}_j^*(t) = \bar{G}_j(t)$ and from equation (5) that $\lambda_j(t) = h_j(t)$. Thus in the independent risks case the $\bar{G}_j^*(t)$ are in fact the marginal survival distributions of the T_j.

Zheng and Klein [30] generalized the result on identifiability in the independent risks case by proving that the marginal distributions are identifiable when the dependence of the T_j is given by a known copula.

3.2 Computing bounds for the marginal survival functions

Peterson [25] gave bounds for the marginal distribution functions $G_j(t) = P(T_j \leq t)$ in terms of the observable sub-distribution functions F_j. The bounds are given by

$$F_j(t) \leq G_j(t) \leq F(t), \tag{7}$$

which are easily verified. Peterson [25] showed, moreover, that these bounds are pointwise sharp. They are not, however, functionally sharp. In fact, Crowder [11] found that the functions $G_j(t) - F_j(t)$ need to be non-decreasing in addition to being non-negative. Subsequently, Bedford and Meilijson [6] obtained the complete characterization of the feasible marginal distribution functions G_j for a given set of sub-distribution functions F_j. They showed in particular that the condition that the functions $G_j(t) - F_j(t)$ are non-negative and non-decreasing, is also sufficient, provided a subtle additional measure theoretic assumption is satisfied. Note that when F_j and G_j are differentiable, this leads to the inequality

$$f_j(t) \leq g_j(t) \text{ for all } t > 0, \tag{8}$$

where $g_j(t)$ is the marginal density function of T_j. We use this inequality in the following example.

Example (adapted from Bedford and Meilijson [6]). Consider the model with $k = 2$ given by constant sub-hazard functions $\lambda_j(t) = \lambda_j$, $j = 1, 2$. In this case $F_j(t) = (\lambda_j/\lambda_+)\,(1 - e^{-\lambda_+ t})$ and $F(t) = 1 - e^{-\lambda_+ t}$, where $\lambda_+ = \lambda_1 + \lambda_2$. Assume now that T_1 has an exponential marginal distribution, so that $G_1(t) = 1 - e^{-\lambda t}$ for some λ. The upper bound of inequality (7) easily gives $\lambda \le \lambda_+$. Further, the inequality (8) with $j = 1$ gives $\lambda_1 e^{-\lambda_+ t} \le \lambda e^{-\lambda t}$ for all $t > 0$, which implies $\lambda \ge \lambda_1$ by letting $t \to 0$. Thus we have shown that any feasible value of λ must satisfy the inequality

$$\lambda_1 \le \lambda \le \lambda_+.$$

Bedford and Meilijson [6] in fact showed that the set of possible values is the half-open interval $[\lambda_1, \lambda_1 + \lambda_2)$. Note that an assumption that T_1 and T_2 are independent leads to $\lambda = \lambda_1$. The example therefore shows that the independence assumption leads to the most optimistic value of the failure rate λ among the ones that are possible when the dependence is not specified. Williams and Lagakos [29] proved a corresponding result in a more general setting.

3.3 Parametric identifiability

Note that the meaning of identifiability of marginal distributions as discussed above has been in the non-parametric sense that the marginal distribution functions G_j can be derived in terms of the sub-distribution functions F_j. If a parametric model is specified for the latent failure times, then the identifiability problem is a completely different one since it now has to do with identification of a finite set of parameters. Crowder [11, Ch. 7.7] and Moeschberger and Klein [23] review models for which identifiability holds (see Section 4.2).

4 MODELLING OF COMPETING RISKS

4.1 Modellling of sub-distributions

Mixture models. These are models given by specifying sub-distribution functions of the form $F_j(t) = \pi_j Q_j(t)$ for given (parametric) distribution functions $Q_j(t)$ and weights π_j with $\sum_{j=1}^{k} \pi_j = 1$. An example is the Weibull version where $\bar{Q}_j(t) = \exp\{-(t/\theta_j)^{\alpha_j}\}$.

Modelling sub-hazard functions. Another common approach is to assume parametric models for the sub-hazard functions, for example using Weibull hazards, $\lambda_j(t; \alpha_j, \theta_j) = (\alpha_j/\theta_j)\,(t/\theta_j)^{\alpha_j - 1}$.

Regression models. Let $\mathbf{x}$ be a vector of covariates for the unit under study. Two main approaches for regression modelling in survival analysis are proportional hazards models and **accelerated life models**. The versions for competing risks can be given as follows.

Proportional hazards: The sub-hazard functions are given by $\lambda_j(t;\mathbf{x}) = \psi_j(\mathbf{x})\lambda_{0j}(t)$ where $\psi_j(\mathbf{x})$ is a positive function of the covariates, for example $\psi_j(\mathbf{x}) = \exp\{\boldsymbol{\beta}'\mathbf{x}\}$ for a parameter vector $\boldsymbol{\beta}$. If $\lambda_{0j}(t) = \lambda_0(t)$ does not depend on j, then T and C are stochastically independent.

Accelerated life model: The dependence of $\mathbf{x}$ is here through factors $\phi_j(\mathbf{x})$ which accelerate time in such a manner that $F_j(t;\mathbf{x}) = F_{0j}(\phi_j(\mathbf{x})t)$.

4.2 Modelling of latent variables

The traditional way of modelling dependent risks has been through specification of the joint distribution function $K(t_1,\ldots,t_k)$ or the joint survival function $\bar{K}(t_1,\ldots,t_k)$. Classical examples are the bivariate normal and Weibull distributions considered by Moeschberger [22] and for which there are no identifiability problems. Other simple examples are given by the following.

Example: Bivariate exponential (Gumbel [15]). Let $k = 2$ and define

$$\bar{K}(t_1,t_2) = \exp\{-\lambda_1 t_1 - \lambda_2 t_2 - \nu t_1 t_2\}.$$

Then $\lambda_j(t) = \lambda_j + \nu t$, so that the independent risks model giving the same sub-distribution functions would have marginal survival functions $\bar{G}_j^*(t) = \exp\{-\lambda_j t - \nu t^2/2\}$. However, the true marginal survival functions corresponding to the given $\bar{K}(t_1,t_2)$ are $\bar{G}_j(t) = \exp\{-\lambda_j t\}$. The dilemma is that it is not possible to distinguish between these two models from observation of (T,C). Note, however, that for both models we have identifiability of all the parameters $\lambda_1, \lambda_2, \nu$.

Example: Preventive maintenance (PM) modelling. In subsection 1.2 we considered the case with $k = 2$ where T_1, T_2 are, respectively, the time of critical failure of a component and the potential time of a PM. Cooke [7], [9] introduced the notion of random signs censoring which is tailored for such cases. In our notation the PM-time T_2 is called a random signs censoring of the failure time T_1 if the event $\{T_2 < T_1\}$ is stochastically independent of T_1. The idea is that the component emits some kind of signal before failure, and that this signal is discovered with a probability which does not depend on the age of the component. The interesting fact is that random signs censoring implies identifiability of the distribution of T_1. Lindqvist, Støve and Langseth [21] suggested a special case of random signs censoring obtained by introducing a so called repair alert function which describes the "alertness" of the maintenance crew as a function of time.

5 STATISTICAL INFERENCE

Consider the case where we have data of the form (T, C) for n independent observation units. In practice such data are often right censored by some source independent of the k given risks. For example, this could be a censoring due to a time limit of the experiment (type I censoring). If the ith observation is non-censored, we observe both the lifetime t_i and the cause c_i. On the other hand, if the ith observation is right censored at time t_i, then we do not observe c_i and all we know is that $T > t_i$. If we let $\delta_i = 0$ if the ith observation is censored and $\delta_i = 1$ otherwise, we get the likelihood function

$$L = \prod_{i=1}^{n} f_{c_i}(t_i)^{\delta_i} \bar{F}(t_i)^{1-\delta_i} = \prod_{i=1}^{n} \lambda_{c_i}(t_i)^{\delta_i} \bar{F}(t_i),$$

where the last equality follows by using equation (1). Further, defining $\delta_{ij} = I(c_i = j)$ when $\delta_i = 1$ and $\delta_{ij} = 0$ when $\delta_i = 0$ and using equation (3), we arrive at (see Lawless [19, Ch. 9.1])

$$L = \prod_{j=1}^{k} \left[\prod_{i=1}^{n} g_j^*(t_i)^{\delta_{ij}} \bar{G}_j^*(t_i)^{1-\delta_{ij}} \right] \equiv \prod_{j=1}^{k} L_j, \tag{9}$$

where $g_j^*(t) = \lambda_j(t)\bar{G}_j^*(t)$ is the density function corresponding to $\bar{G}_j^*(t)$.

Thus we can write L as a product of the likelihoods L_j, where L_j has the same form as the likelihood function of a censored sample associated with the jth failure cause where all observations where C is not observed to equal j are treated as censorings.

Any of the above expressions for the likelihood L may be used for parametric **maximum likelihood estimation**, depending on the way the model is defined. Equation (9) has important implications for non-parametric estimation. In fact, the likelihood L_j is of the same form as the one leading to the usual Kaplan-Meier estimator (see [2]) of a survival function under independent right censoring. This means in particular that the "artificial" survival functions $\bar{G}_j^*(t)$ and hence $\Lambda_j(t) = -\log \bar{G}_j^*(t)$, can be estimated non-parametrically by the Kaplan-Meier estimator. Alternatively, one may from the same reasoning estimate the Λ_j for each j using the Nelson-Aalen estimator ([2]). In this case, denoting the resulting estimator by $\hat{\Lambda}_j(t)$, it follows from equation (2) that one may estimate the F_j non-parametrically by

$$\hat{F}_j(t) = \int_0^t \hat{\bar{F}}(u)\mathrm{d}\hat{\Lambda}_j(u),\ j = 1, ..., k, .$$

where $\hat{\bar{F}}(t)$ is the ordinary Kaplan-Meier estimate of the survival function $\bar{F}(t)$ of T. Note that this estimator uses the censored data obtained by collapsing the k failure causes into one single cause.

6 BEYOND CLASSICAL COMPETING RISKS

Markov process models. Aalen [1] modelled the classical competing risks problem as a continuous-time **Markov process** with one working state ("0") and k absorbing failure states corresponding to the k risks. More generally one may consider Markov processes with more complex state spaces and again k absorbing states corresponding to the k failure types. This leads to the consideration of multivariate phase type distributions, see Assaf et al. [3]. Hokstad and Frøvig [16] considered failure models for periodically tested items where the failures are either degraded or critical. Whitmore [28] discussed the use of first-passage-time distributions connected with multidimensional **Brownian motion** as models for competing risks.

Competing risks in repairable systems. Consider a system where failures are classified in k different types. Until now we have considered the case of non-repairable units observed until failure or PM. Now assume that the unit is repaired after failure, then is put into operation again, and that the process continues in this way (see eqr 356). Suppose that, at each failure, the type of event is recorded. Assume also that time durations of repair and maintenance can be disregarded, so that the system is always restarted immediately after failure or maintenance action This leads to the observation of a marked point process $(S_1, C_1), (S_2, C_2), \ldots$ with successive failure times $0 < S_1 < S_2 < \cdots$ and marks C_i in $\{1, 2, \ldots, k\}$. The properties of such a process depends on the repair strategy. For example, a perfect repair (renewal) of the system at failures leads to independent and identically distributed pairs (T_i, C_i) where $T_i = S_i - S_{i-1}$ $(i = 1, 2, \ldots)$ and $S_0 = 0$. We are hence back to the basic case considered in this article. In general, however, the situation could be more complicated. Doyen and Gaudoin [13] present a point process approach for modelling of imperfect repair in competing risks situations between failure and PM. Bedford and Lindqvist [5] considered a series system of k repairable components where only the failing component is repaired at failures. A general setup for this kind of processes is suggested by Lindqvist [20].

7 DISCUSSION

As argued in the article, the latent failure time approach to competing risks is particularly useful in reliability applications. However, as risks will usually be dependent, the identifiability problem occurs. Recall that although additional assumptions on the dependence may lead to identifiability, these assumptions are non-testable from data of the form (T, C). Prentice et al. [26] therefore rejected the latent failure time approach and instead advocated the use of the observable sub-hazard functions in analyses of competing risks. One may argue against this, however, that in practical applications it often makes good sense to use ones information and prior beliefs in order to model the underlying probability mechanisms beyond what is actually observable.

On the other hand, an erroneous assumption of independent risks may lead to seriously misleading conclusions. As noted by Cooke [8], such assumptions are often made in the assessments of competing failure rates in reliability databases. One then assumes that for each of the k failure causes, failures occur as independent **Poisson processes**. This implies, however, that the rate of occurrence of each of the competing risks would be unaffected by removing the others. For competing risks corresponding to critical failure and PM this means that the rate of occurrence of critical failures would be unaffected by stopping preventive maintenance activity, an assumption which is completely unreasonable. The appropriate method would be to invoke a more careful modelling by competing risks, using if possible all available information. More sophisticated methods for analysis of competing risks in connection with repairable systems, as briefly mentioned in the previous section, may be needed here.

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