CHAPTER 1

A MAINTENANCE MODEL FOR COMPONENTS EXPOSED TO SEVERAL FAILURE MECHANISMS AND IMPERFECT REPAIR

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We investigate the mathematical modelling of maintenance and repair of components that can fail due to a variety of failure mechanisms. Our motivation is to build a model, which can be used to unveil aspects of the quality of the maintenance performed. The model we propose is motivated by imperfect repair models, but extended to model preventive maintenance as one of several "competing risks". This helps us to avoid problems of identifiability previously reported in connection with imperfect repair models.

1. Introduction

In this chapter we employ a model for components which fail due to one of a series of "competing" failure mechanisms, each acting independently on the system. The components under consideration are repaired upon failure, but are also preventively maintained. The preventive maintenance (PM) is performed periodically with some fixed period τ, but PM can also be performed out of schedule due to casual observation of an evolving failure. The maintenance need not be perfect; we use a modified version of the imperfect repair model by Brown and Proschan[4] to allow a flexible yet simple maintenance model. Our motivation for this model is to estimate quantities which describe the "goodness" of the maintenance crew; their ability to prevent failures by performing thorough maintenance at the correct time. The data required to estimate the parameters in the model we propose are the intermediate failure times, the "winning" failure mechanism associated with each failure (i.e. the failure mechanism leading to the failure), as well as the maintenance activity. This data is found in most modern reliability data banks.

The rest of this chapter is outlined as follows: We start in Section 2 with the problem definition by introducing the type of data and parameters we consider. Next, the required theoretical background is sketched in Section 3, followed by a complete description of the proposed model in Section 4. We make some concluding remarks in Section 5.

2. Problem Definition, Typical Data and Model Parameters

Consider a mechanical component which may fail at random times, and which after failure is immediately repaired and put back into service. In practice there can be several root causes for the failure, e.g. vibration, corrosion, etc. We call these causes failure mechanisms and denote them by $M_1, \dots, M_k$. It is assumed that each failure can be classified as the consequence of exactly one failure mechanism.

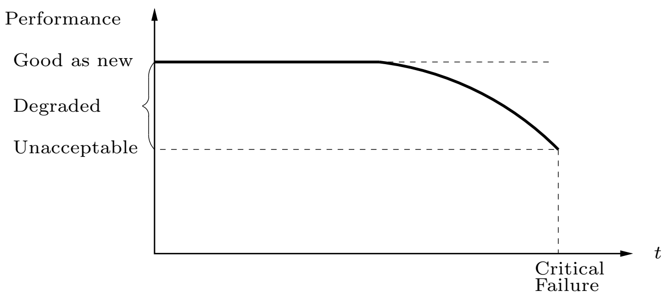


Fig. 1. Component with degrading performance.

The component is assumed to undergo preventive maintenance (PM), usually at fixed time periods $\tau > 0$. In addition, the maintenance crew may perform unscheduled preventive maintenance of a component if required. The rationale for unscheduled PM is illustrated in Fig. 1: We assume that the component is continuously deteriorating when used, so that the performance gradually degrades until it falls outside a preset acceptable margin. As soon as the performance is unacceptable, we say that the component experiences a critical failure. Before the component fails it may exhibit inferior but admissible performance. This is a "signal" to the maintenance crew that a critical failure is approaching, and that the inferior component may be repaired. When the maintenance crew intervenes and repairs a component before it fails critically, we call it a degraded failure, and the repair action is called (an unscheduled) preventive maintenance. On the other hand, the repair activity performed after a critical failure is called a corrective maintenance.

The history of the component may in practice be logged as shown in Table 1. The events experienced by the component can be categorized as either (*i*) Critical failures, (*ii*) Degraded failures, or (*iii*) External events (component taken out of service, periodic PM, or other kind of censoring).

Table 1. Example of data describing the history of a fictitious component.

Time	Event	Failure mech.	Severity
0	Put into service	—	—
314	Failure	Vibration	Critical
8.760	(Periodic) PM	External	—
17.520	(Periodic) PM	External	—
18.314	Failure	Corrosion	Degraded
20.123	Taken out of service	External	—

The data for a single component can now formally be given as an ordered sequence of points

$$(Y_i, K_i, J_i);\ i = 1, 2, \ldots, n, \tag{1}$$

where each point represents an event. Here

$$\begin{aligned}
Y_i = &\ \text{inter-event time, i.e. time since previous event} \\
&\ \text{(time since start of service if } i = 1) \\
K_i = &\begin{cases} m \text{ if failure mechanism } M_m\ (m = 1, \ldots, k) \\ 0 \text{ if external event} \end{cases} \\
J_i = &\begin{cases} 0 \text{ if critical failure} \\ 1 \text{ if degraded failure} \\ 2 \text{ if external event.} \end{cases}
\end{aligned} \tag{2}$$

The data in Table 1 can thus be coded as (with $M_1 =$ Vibration, $M_2 =$ Corrosion),

$$(314, 1, 0),\ (8446, 0, 2),\ (8760, 0, 2),\ (794, 2, 1),\ (1809, 0, 2).$$

A complete set of data will typically involve events from several similar components. The data can then be represented as

$$(Y_{ij}, K_{ij}, J_{ij});\ i = 1, 2, \ldots, n_j;\ j = 1, \ldots, r, \tag{3}$$

where j is the index which labels the r component.

In practice there may also be observed covariates with such data. The models considered in this chapter will, however, not include this possibility even though they could easily be modified to do so.

Our aim is to present a model for data of type (1) (or (3)). The basic ingredients in such a model are the hazard rates $\omega_m(t)$ at time t for each failure mechanism M_m, for a component which is new at time $t = 0$. We assume that $\omega_m(t)$ is a continuous and integrable function on $[0, \infty)$. In practice it will be important to estimate $\omega_m(\cdot)$ since this information may, e.g., be used to plan future maintenance strategies.

The most frequently used models for repairable systems assume either perfect repair (renewal process models) or minimal repair (nonhomogeneous Poisson-process models). Often none of these may be appropriate, and we shall here adopt the idea of the imperfect repair model presented by Brown and Proschan[4]. This will introduce two parameters per failure mechanism:

$$\begin{aligned} p_m = &\text{ probability of perfect repair for a preventive maintenance of } M_m \\ \pi_m = &\text{ probability of perfect repair for a corrective maintenance of } M_m. \end{aligned}$$

These quantities are of interest since they can be used as indications of the quality of maintenance. The parameters may in practice be compared between plants and companies, and thereby unveil maintenance improvement potential.

Finally, our model will take into account the relation between preventive and corrective maintenance. It is assumed that the component gives some kind of "signal", which will alert the maintenance crew to perform a preventive maintenance before a critical failure occurs. Thus it is not reasonable to model the (potential) times for preventive and corrective maintenance as stochastically independent. We shall therefore adopt the random signs censoring of Cooke[5]. This will eventually introduce a single new parameter q_m for each failure mechanism, with interpretation as the probability that a critical failure is avoided by a preceding unscheduled preventive maintenance.

In the cases where there is a single failure mechanism, we shall drop the index m on the parameters above.

3. Basic Ingredients of the Model

In this section we describe and discuss the two main building blocks of our final model. In Section 3.1 we consider the concept of imperfect repair, as defined by Brown and Proschan[4]. Then in Section 3.2 we introduce our basic model for the relation between preventive and corrective maintenance. Throughout the section we assume that there is a single failure mechanism ($k = 1$).

3.1. *Imperfect Repair*

Our point of departure is the imperfect repair model of Brown and Proschan[4], which we shall denote BP in the following. Consider a single sequence of failures, occurring at successive times $T_1, T_2, \ldots$ As in the previous section we let the Y_i be times between events, see Fig. 2. Furthermore, $N(t)$ is the number of events in $(0, t]$, and $N(t^-)$ is the number of events in $(0, t)$.

For the explanation of imperfect repair models it is convenient to use the conditional intensity

$$\lambda(t \mid \mathcal{F}_{t^-}) = \lim_{\Delta t \downarrow 0} \frac{P(\text{event in } [t, t + \Delta t) \mid \mathcal{F}_{t^-})}{\Delta t},$$

where $\mathcal{F}_{t^-}$is the history of the counting process[2] up to time t. This notation enables us to review some standard repair models. Let $\omega(t)$ be the hazard rate of a component of "age" t. Then perfect repair is modelled by $\lambda\left(t \mid \mathcal{F}_{t^-}\right) = \omega\left(t - T_{N(t^-)}\right)$ which means that the age of the component at time t equals $t - T_{N(t^-)}$, the time elapsed since the last event. Minimal repair is modelled by $\lambda\left(t \mid \mathcal{F}_{t^-}\right) = \omega\left(t\right)$, which means that the age at any time t equals the calendar time t. Imperfect repair can be modelled by $\lambda\left(t \mid \mathcal{F}_{t^-}\right) = \omega\left(\Xi_{N(t^-)} + t - T_{N(t^-)}\right)$ where $0 \leq \Xi_i \leq T_i$ is some measure of the effective age of the component immediately after the ith event, more precisely, immediately after the corresponding repair. In the BP model, Ξ_i is defined indirectly by letting a failed component be given perfect repair with probability p, and minimal repair with probability $1 - p$.

For simplicity of notation we follow Kijima[8] and introduce random variables D_i to denote the outcome of the repair immediately after the ith event. If we put $D_i = 0$ for a perfect repair and $D_i = 1$ for a minimal one, it follows that

$$\Xi_i = \sum_{j=1}^{i} \left(\prod_{k=j}^{i} D_k \right) Y_j.$$

The BP model with parameter p corresponds to assuming that the D_i are *i.i.d.* and independent of $Y_1, Y_2, \ldots$, with $P(D_i = 0) = p$, $P(D_i = 1) = 1-p$, $i = 1, \ldots, n$.

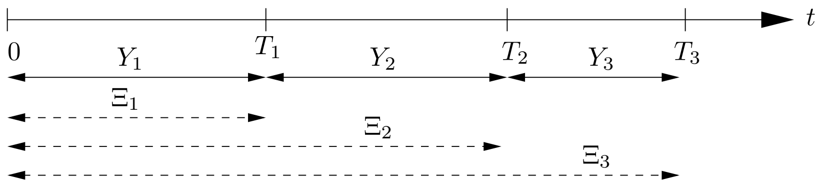


Fig. 2. In imperfect repair models there are three time dimensions to measure the age of a component: Age versus calendar time T_i, age versus inter-event times Y_i, and effective age Ξ_i. The values of Ξ_i, $i > 1$, depend upon both inter-event times and maintenance history. This is indicated by dotted lines for the Ξ_i.

BP type models have been considered by several authors, including Block *et al.*[3] who extended the model to allow the parameter p to be time varying, Kijima[8] who studied two general repair models for which BP is a special case, Hollander *et al.*[7] who studied statistical inference in the model, Dorado *et al.*[6] who proposed a more general model with BP as a special case, and most notably for the present work, Whitaker and Samaniego[10] whose results we discuss in further detail below.

Whitaker and Samaniego[10] found non-parametric maximum likelihood estimators for (p, F) in the BP model, where F is the distribution function corresponding to the hazard $\omega(\cdot)$. They noted that p is in general not identifiable if only the inter-event times Y_i are observed. The problem is related to the memoryless property of the exponential distribution, and is hardly a surprise. To ensure identifiability, Whitaker and Samaniego made strong assumptions about data availability, namely that the type of repair (minimal or perfect) is reported for each repair action (i.e., the variables D_j are actually observed). In real applications, however, exact information on the type of repair is rarely available. As we shall see in Section 4.2, identifiability of p is still possible in the model by appropriately modelling the maintenance actions.

In order to illustrate estimation in the BP model based on the Y_i alone, we consider the failure times of Plane 7914 from the air conditioner data of Proschan[9] given in Table 2. These data were also used by Whitaker and Samaniego[10]. The joint density of the observations $Y_1, \ldots, Y_n$ can be calculated as a product of conditional densities,

$$f(y_1, \ldots, y_n) = f(y_1) f(y_2|y_1) \cdots f(y_n|y_1, \ldots, y_{n-1}).$$

For computation of the ith factor we condition on the unobserved

Table 2. Proschan's air conditioner data; inter-event times of plane 7914.

50	44	102	72	22	39	3	15
197	188	79	88	46	5	5	36
22	139	210	97	30	23	13	14

$D_1, \ldots, D_{i-1}$, getting

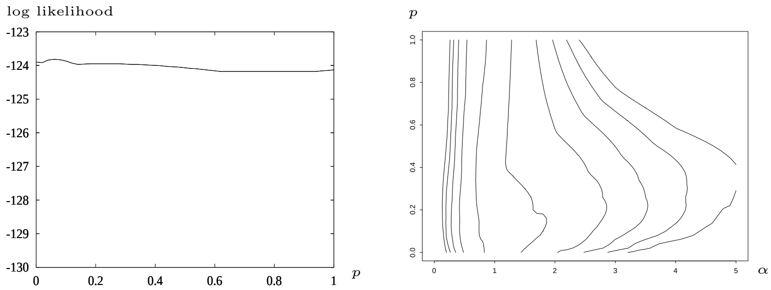
$$\begin{aligned} f(y_i \mid y_1, \ldots, y_{i-1}) &= \sum_{d_1,\ldots,d_{i-1}} f(y_i \mid y_1, \ldots, y_{i-1}, d_1, \ldots, d_{i-1}) \\ &\quad \times f(d_1, \ldots, d_{i-1} \mid y_1, \ldots, y_{i-1}) \\ &= \sum_{j=1}^{i} f(y_i \mid y_1, \ldots, y_{i-1}, d_{j-1} = 0, d_j = \cdots = d_{i-1} = 1) \\ &\quad \times P(D_{j-1} = 0, D_j = \cdots = D_{i-1} = 1) \\ &= \sum_{j=1}^{i} \omega\left(\sum_{k=j}^{i} y_k\right) e^{-\left[\Omega\left(\sum_{k=j}^{i} y_k\right) - \Omega\left(\sum_{k=j}^{i-1} y_k\right)\right]} (1-p)^{i-j}\, p^{\delta(j>1)}, \end{aligned}$$

where $\Omega(x) = \int_0^x \omega(t)dt$ is the cumulative hazard function and $\delta(j > 1)$ is 1 if $j > 1$ and 0 otherwise. The idea is to partition the set of vectors $(d_1, \ldots, d_{i-1})$ according to the number of 1s immediately preceding the ith event.

Let the cumulative hazard be given by $\Omega(x) = \mu x^\alpha$ for unknown μ and α. The profile log likelihoods of the single parameter p and the pair (α, p) are shown in Fig. 3a) and Fig. 3b) respectively. The maximum likelihood estimates are $\hat{\alpha} = 1.09$, $\hat{\mu} = \exp(-4.81)$, and $\hat{p} = 0.01$. However, the data contain very little information about p; this is illustrated in Fig. 3a). It is seen that both $p = 0$, corresponding to an NHPP, and $p = 1$, corresponding to a Weibull renewal process are "equally" possible models here. The problem is closely connected to the problem of unidentifiability of p, noting that the maximum likelihood estimate of α is close to 1. Indeed, the exponential model with $\alpha = 1$ fixed gives the maximum log likelihood -123.86 while the maximum value in the full model (including μ, α and p) is only marginally larger, -123.78.

3.2. *Modelling Preventive versus Corrective Maintenance*

Recall from Section 2 that PM interventions are basically periodic with some fixed period τ, but that unscheduled preventive maintenance may still be performed within a PM period, reported as degraded failures. Thus



a) Profile log likelihood of p b) Profile log likelihood of (α, p)

Fig. 3. Profile log likelihoods for the data in Table 2. Fig. 3a) shows the profile likelihood of p, Fig. 3b) shows the (α, p)-profile likelihood.

degraded failures may censor critical failures, and the two types of failure may be highly correlated.

A number of possible ways to model interaction between degraded and critical failures are discussed by Cooke[5]. We adopt one of these, called random signs censoring. In the notation introduced in Section 2 we consider here the case when we observe pairs (Y_i, J_i) where the Y_i are inter-event times whereas the J_i are indicators of failure type (critical or degraded). For a typical pair (Y, J) we let Y be the minimum of the potential critical failure time X and the potential degraded failure time Z, while $J = I(Z < X)$ is the indicator of the event $\{Z < X\}$ (assuming that $P(Z = X) = 0$ and that there are no external events). Thus we have a competing risk problem. However, while X and Z would traditionally be treated as independent, random signs censoring makes them dependent in a special way.

The basic assumption of random signs censoring is that the event of successful preventive maintenance, $\{Z < X\}$, is stochastically independent of the potential critical failure time X. In other words, the conditional probability $q(x) = P(Z < X | X = x)$ does not depend on the value of x.

Let X have hazard rate function $\omega(x)$ and cumulative hazard $\Omega(x)$. In addition to the assumption of random signs censoring, we will assume that conditionally, given $Z < X$ and $X = x$, the distribution of the intervention time Z satisfies

$$P(Z \leq z \mid X = x, Z < X) = \frac{\Omega(z)}{\Omega(x)}, \ 0 \leq z \leq x. \tag{4}$$

To see why (4) is reasonable, consider Fig. 4. When "Nature" has chosen in favour of the crew and has selected the time to critical failure,

$X = x$, which the crew will have to beat, she first draws a value u uniformly from $[0, \Omega(x)]$. Then the time for preventive maintenance is chosen as $Z = \Omega^{-1}(u)$, where $\Omega^{-1}(\cdot)$ is the inverse function of $\Omega(\cdot)$. Following this procedure makes the conditional density of Z proportional to the intensity of the underlying failure process. This seems like a coarse but somewhat reasonable description of the behaviour of a competent maintenance crew.

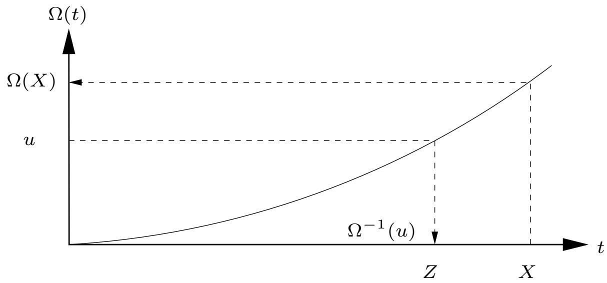


Fig. 4. Time to PM conditioned on $\{Z < X, X = x\}$.

Our joint model for (X, Z) is thus defined from the following:

(i) X has hazard rate $\omega(\cdot)$.
(ii) $\{Z < X\}$ and X are stochastically independent.
(iii) Z given $Z < X$ and $X = x$ has distribution function (4).

These requirements determine the distribution of the observed pair (Y, J) as follows. First, by (ii) we get

$$\begin{aligned} P(y \le Y \le y + dy, J = 0) &= P(y \le X \le y + dy, X < Z) \\ &= (1 - q)\, \omega(y)\, \exp(-\Omega(y))\, dy \end{aligned}$$

where we introduce the parameter $q = P(Z < X)$. Next,

$$\begin{aligned} &P(y \le Y \le y + dy, J = 1) \\ &\quad = P(y \le Z \le y + dy, Z < X) \\ &\quad = \int_y^\infty P(y \le Z \le y + dy | X = x, Z < X) \\ &\qquad\qquad \times P(Z < X | X = x)\, \omega(x) \exp(-\Omega(x))\, dx \\ &\quad = q\, \omega(y)\, dy \int_y^\infty \omega(x)\, \exp(-\Omega(x)) \,/\, \Omega(x)\, dx \\ &\quad = q\, \omega(y)\, \mathrm{Ie}(\Omega(y))\, dy, \end{aligned}$$

where $\mathrm{Ie}(t) = \int_t^\infty \exp(-u)/u\,du$ is known as the exponential integral[1].

It is now straightforward to establish the density and distribution function of Y,

$$f_Y(y) = (1-q)\,\omega(y)\,\exp\left(-\Omega(y)\right) + q\,\omega(y)\,\mathrm{Ie}(\Omega(y)) \tag{5}$$

and

$$F_Y(y) = P(Y \le y) = 1 - \exp(-\Omega(y)) + q\,\Omega(y)\,\mathrm{Ie}(\Omega(y)). \tag{6}$$

Note that the proposed maintenance model introduces only one new parameter, namely q.

The distribution (5) for Y is a mixture distribution, with one component representing the failure distribution one would have without preventive maintenance, and the other mixture component being the conditional density of time for PM given that PM "beats" critical failure. It is worth noticing that the distribution with density $\omega(y)\,\mathrm{Ie}(\Omega(y))$ is stochastically smaller than the distribution with density $\omega(y)\,\exp(-\Omega(y))$; this is a general consequence of random signs censoring.

4. General Model

Recall that the events in our most general setting are either critical failures, degraded failures or external events; consider Fig. 2. We shall assume that corrective maintenance is always performed following a critical failure, while preventive maintenance is performed both after degraded failures and external events. Moreover, in the case of several failure mechanisms, any failure is treated as an external event for all failure mechanisms except the one failing.

4.1. *Single Failure Mechanism*

In this case the data for one component are $(Y_i, J_i);\ i = 1, \ldots, n$ with J_i now defined as in (2) with three possible values. Suppose for a moment that all repairs, both corrective and preventive, are perfect. Then we shall assume that the (Y_i, J_i) are *i.i.d.* observations of (Y, J) where $Y = \min(X, Z, U)$, (X, Z) is distributed as in Section 3.2, and U is the (potential) time of an external event. The U is assumed to be stochastically independent of (X, Z) and to have a distribution which does not depend on the parameters of our model. It follows that we can disregard the terms corresponding to U in the likelihood calculation. The likelihood contribution from an observation

(Y, J) will therefore be as follows (see section 3.2):

$$\begin{aligned} f(y,0) &= (1-q)\,\omega(y)\,\exp\left(-\Omega(y)\right) \\ f(y,1) &= q\,\omega(y)\,\mathrm{Ie}(\Omega(y)) \\ f(y,2) &= \exp\left(-\Omega(y)\right) - q\,\Omega(y)\,\mathrm{Ie}(\Omega(y)). \end{aligned} \tag{7}$$

The last expression follows from (6) and corresponds to the case where all we know is that $\max(X, Z) > y$.

To the model given above we now add imperfect repair. Recall that in the BP model there is a probability p of perfect repair ($D_i = 0$) after each event. We shall here distinguish between preventive maintenance and corrective maintenance by letting D_i equal 0 with probability p if the ith event is a preventive maintenance or an external event, and with probability π if the ith event is a critical failure. Moreover, we shall assume that for all i we have $D_1, \ldots, D_i$ conditionally independent given $y_1, \ldots, y_i, j_1, \ldots, j_i$.

From this we are able to write down the likelihood of the data as a product of the following conditional distributions. The derivation is a straightforward extension of the one in Section 3.1.

$$\begin{aligned} & f\Big((y_i, j_i) \mid (y_1, j_1), \ldots, (y_{i-1}, j_{i-1})\Big) \\ &\quad = \sum_{d_1,\ldots,d_{i-1}} f((y_i, j_i) \mid (y_1, j_1), \ldots, (y_{i-1}, j_{i-1}), d_1, \ldots, d_{i-1}) \\ &\qquad\qquad \times f(d_1, \ldots, d_{i-1} | (y_1, j_1), \ldots, (y_{i-1}, j_{i-1})) \\ &\quad = \sum_{j=1}^{i} f\left((y_i, j_i) \middle| \xi_{i-1} = \sum_{k=j}^{i-1} y_k\right) \\ &\qquad\qquad \times P\left(D_{j-1} = 0, D_j = \cdots = D_{i-1} = 1 | j_1, \ldots, j_{i-1}\right). \end{aligned}$$

Here $P(D_{j-1} = 0, D_j = \cdots = D_{i-1} = 1 \mid j_1, \ldots, j_{i-1})$ is a simple function of p and π. Thus, what remains to be defined are the conditional densities $f\left((y_i, j_i)|\xi_{i-1}\right)$, i.e. the conditional densities of (Y_i, J_i) given that the age of the component immediately after the $(i-1)$th event is ξ_{i-1}. We shall define these to equal the conditional densities given no event in $(0, \xi_{i-1})$, of the distribution given in (7). Thus we have

$$\begin{aligned} f\left((y_i, 0) \mid \xi_{i-1}\right) &= \frac{(1-q)\,\omega(\xi_{i-1} + y_i)\,\exp(-(\Omega(\xi_{i-1} + y_i)))}{\exp(-\Omega(\xi_{i-1})) - q\,\Omega(\xi_{i-1})\,\mathrm{Ie}(\Omega(\xi_{i-1}))} \\ f\left((y_i, 1) \mid \xi_{i-1}\right) &= \frac{q\,\omega(\xi_{i-1} + y_i)\,\mathrm{Ie}(\Omega(\xi_{i-1} + y_i))}{\exp(-\Omega(\xi_{i-1})) - q\,\Omega(\xi_{i-1})\,\mathrm{Ie}(\Omega(\xi_{i-1}))} \\ f\left((y_i, 2) \mid \xi_{i-1}\right) &= \frac{\exp(-\Omega(\xi_{i-1} + y_i)) - q\,\Omega(\xi_{i-1} + y_i)\,\mathrm{Ie}(\Omega(\xi_{i-1} + y_i))}{\exp(-\Omega(\xi_{i-1})) - q\,\Omega(\xi_{i-1})\,\mathrm{Ie}(\Omega(\xi_{i-1}))}. \end{aligned}$$

If we have data from several independent components, the complete likelihood is given as the product of the individual likelihoods.

4.2. *Identifiability of Parameters*

The present discussion of identifiability is inspired by the corresponding discussion by Whitaker and Samaniego[10], who considered the simple BP model.

Refer again to the model of the previous subsection. We assume here that, conditional on $(Y_1, J_1), (Y_2, J_2), \ldots, (Y_{i-1}, J_{i-1})$, the (potential) time to the next external event is a random variable U with continuous distribution G and support on all of $(0, \tau]$ where τ as before is the regular maintenance interval. Moreover, the distribution G does not depend on the parameters of the model, and it is kept fixed in the following.

We also assume that $\omega(x) > 0$ for all $x > 0$ and that $0 < q < 1$. The parameters of the model are ω, q, p, π. These, together with G, determine a distribution of $(Y_1, J_1), \ldots, (Y_n, J_n)$ which we call $\mathcal{F}_{(\omega,q,p,\pi)}$. Here n is kept fixed.

The question of identifiability can be put as follows: Suppose

$$\mathcal{F}_{(\omega,q,p,\pi)} = \mathcal{F}_{(\omega^*,q^*,p^*,\pi^*)}, \tag{8}$$

which means that the two parameterizations lead to the same distribution of the observations $(Y_1, J_1), \ldots, (Y_n, J_n)$. Can we from this conclude that $\omega = \omega^*$, $q = q^*$, $p = p^*$, $\pi = \pi^*$?

First note that (8) implies that the distribution of (Y_1, J_1) is the same under the two parameterizations; $Y_1 = \min(X, Z, U)$. It is clear that each of the following two types of probabilities are the same under the two parameterizations,

$$P(x \le X \le x + dx, Z > x, U > x)$$
$$P(z \le Z \le z + dz, X > z, U > z).$$

By independence of (X, Z) and U, and since $P(U > x) > 0$ if and only if $x < \tau$, we conclude that each of the following two types of probabilities are equal under the two parameterizations,

$$P(x \le X \le x + dx, Z > x);\ x < \tau$$
$$P(z \le Z \le z + dz, X > z);\ z < \tau.$$

These probabilities can be written respectively

$$(1 - q)\,\omega(x)\,e^{-\Omega(x)}\,dx;\ x < \tau$$
$$q\,\omega(z)\,\mathrm{Ie}(\Omega(z))\,dz;\ z < \tau.$$

Thus, by integrating from 0 to x we conclude that (8) implies for $x \leq \tau$

$$(1-q)\left(1-e^{-\Omega(x)}\right) = (1-q^*)\left(1-e^{-\Omega^*(x)}\right) \tag{9}$$

$$q\left(1-e^{-\Omega(x)}+\Omega(x)\mathrm{Ie}(\Omega(x))\right) = q^*\left(1-e^{-\Omega^*(x)}+\Omega^*(x)\mathrm{Ie}(\Omega^*(x))\right). \tag{10}$$

We shall now see that this implies that $q = q^*$ and $\Omega(x) = \Omega^*(x)$ for all $x \leq \tau$. Suppose, for contradiction, that there is an $x_0 \leq \tau$ such that $\Omega(x_0) < \Omega^*(x_0)$. Then since both $1-\exp(-t)$ and $1-\exp(-t)+t\,\mathrm{Ie}(t)$ are strictly increasing in t, it follows from respectively (9) and (10) that $1-q > 1-q^*$ and $q > q^*$. But this is a contradiction. In the same manner we get a contradiction if $\Omega(x_0) > \Omega^*(x_0)$. Thus $\Omega(x) = \Omega^*(x)$ for all $x \leq \tau$ (so $\omega(x) = \omega^*(x)$ for all $x \leq \tau$) and hence also $q = q^*$.

We shall see below that in fact we have $\Omega(x) = \Omega^*(x)$ on the interval $(0, n\tau)$, but first we shall consider the identifiability of p and π. For this end we consider the joint distribution of $(Y_1, J_1), (Y_2, J_2)$. In the same way as already demonstrated we can disregard U in the discussion, by independence, but we need to restrict y_1, y_2 so that $y_1 + y_2 \leq \tau$. First, look at

$$\begin{aligned} &P\Big(y_1 \leq Y_1 \leq y_1 + dy_1, J_1 = 0, y_2 \leq Y_2 \leq y_2 + dy_2, J_2 = 0\Big) \\ &\quad = (1-q)\,\omega(y_1)\,e^{-\Omega(y_1)}\Big[\pi(1-q)\omega(y_2)e^{-\Omega(y_2)} \\ &\quad +(1-\pi)\,(1-q)\,\frac{\omega(y_1+y_2)\exp(-\Omega(y_1+y_2))}{\exp(-\Omega(y_1)) - q\,\Omega(y_1)\,\mathrm{Ie}(\Omega(y_1))}\Big]\,dy_1\,dy_2. \end{aligned} \tag{11}$$

This is a linear function of π with coefficient of π proportional to

$$\omega(y_2)\exp(-\Omega(y_2)) - \frac{\omega(y_1+y_2)\exp(-\Omega(y_1+y_2))}{\exp(-\Omega(y_1)) - q\,\Omega(y_1)\,\mathrm{Ie}(\Omega(y_1))}. \tag{12}$$

Using the assumption that $0 < q < 1$ we thus conclude that $\pi = \pi^*$ unless (12) equals 0 for all y_1 and y_2 with $y_1 + y_2 \leq \tau$. Making the similar computation, putting $J_2 = 1$ instead of $J_2 = 0$ in (11), we can similarly conclude that $\pi = \pi^*$ unless

$$\omega(y_2)\mathrm{Ie}(\Omega(y_2)) - \frac{\omega(y_1+y_2)\,\mathrm{Ie}(\Omega(y_1+y_2))}{\exp(-\Omega(y_1)) - q\,\Omega(y_1)\,\mathrm{Ie}(\Omega(y_1))} \tag{13}$$

equals 0 for all y_1 and y_2 with $y_1 + y_2 \leq \tau$. Now, if both (12) and (13) were 0 for all y_1 and y_2 with $y_1 + y_2 \leq \tau$, then we would necessarily have

$$\frac{\exp(-\Omega(y_2))}{\mathrm{Ie}(\Omega(y_2))} = \frac{\exp(-\Omega(y_1+y_2))}{\mathrm{Ie}(\Omega(y_1+y_2))} \tag{14}$$

for all y_1 and y_2 with $y_1 + y_2 \leq \tau$. Since we have assumed that $\omega(\cdot)$ is strictly positive, (14) would imply that $\exp(-t)/\mathrm{Ie}(t)$ is constant for t in some interval (a, b). This is of course impossible by the definition of $\mathrm{Ie}(\cdot)$, and it follows that not both of (12) and (13) can be identically zero. Hence π is identifiable.

Identifiability of p is concluded in the same way by putting $J_1 = 1$ instead of $J_1 = 0$ in (11).

So far we have concluded equality of the parameters q, p, π under the two parameterizations, while we have concluded that $\Omega(x) = \Omega^*(x)$ for all $x \leq \tau$. But then, putting $y_1 = \tau$ in (11), while letting y_2 run from 0 to τ, it follows that $\Omega(x) = \Omega^*(x)$ also for all $\tau < x \leq 2\tau$. By continuing we can eventually conclude that $\Omega(x) = \Omega^*(x)$ for all $0 < x \leq n\tau$.

If $\tau = \infty$, then of course the whole function $\omega(\cdot)$ is identifiable. However, even if $\tau < \infty$ we may have identifiability of all of $\omega(\cdot)$. For example, suppose $\Omega(x) = \mu x^\alpha$ with μ, α positive parameters. Then the parameters are identifiable since (9) in this case implies that $\mu x^\alpha = \mu^* x^{\alpha^*}$ for all $x \leq \tau$. This clearly implies the pairwise equality of the parameters.

4.3. *Several Failure Mechanisms*

We now look at how to extend the model of Section 4.2 to $k > 1$ failure mechanisms and data given as in (1) or (3).

Our basic assumption is that the different failure mechanisms $M_1, \ldots, M_k$ act independently on the component. More precisely we let the complete likelihood for the data be given as the product of the likelihoods for each failure mechanism. Note that the set of events is the same for all failure mechanisms, and that failure due to one failure mechanism is treated as an external event for the other failure mechanisms.

The above assumption implies a kind of independence of the maintenance for each failure mechanism. Essentially we assume that the pairs (X, Z) are independent across failure mechanisms. This is appropriate if there are different maintenance crews connected to each failure mechanisms, or could otherwise mean that the "signals" of degradation emitted from the component are independent across failure mechanisms.

Another way of interpreting our assumption is that, conditional on

$$(y_1, k_1, j_1), \ldots, (y_{i-1}, k_{i-1}, j_{i-1})$$

the next vector (Y_i, K_i, J_i) corresponds to a competing risk situation involving m independent risks, one for each failure mechanism, and each with properties as for the model given in Section 4.1.

The parameters (ω, q, p, π) may (and will) in general depend on the failure mechanism. As regards identifiability of parameters, this will follow from the results for single failure mechanisms of Section 4.2 by the assumed independence of failure mechanisms.

If we have data from several independent components of the same kind, given as in (3), then the complete likelihood is given as the product of the likelihoods for each component.

5. Concluding Remarks

In this chapter we have proposed a simple but flexible model for maintained components which are subject to a variety of failure mechanisms. The proposed model has the standard models of perfect and minimal repair as special cases. Moreover, some of the parameters we estimate (namely p_m, π_m and q_m) can be used to examine the sufficiency of these smaller models. "Small" values of $\hat{q}_m$ accompanied by "extreme" values of all $\hat{p}_m$ and $\hat{\pi}_m$ (either "close" to one or zero) indicate that reduced models are detailed enough to capture the main effects in the data. Making specific model assumptions regarding the preventive maintenance we are able to prove identifiability of all parameters.

Our motivation has been to build a model that could be used to estimate the effect of maintenance, where "effect" has been connected to the model parameters q_m, p_m and π_m. For a given level of maintenance activity, we can use q_m as an indication of the crew's efficiency; their ability to perform maintenance at the correct times to try to stop evolving failures. The p_m and π_m indicate the crew's thoroughness; their ability to actually stop the failure development. The proposed model indirectly estimates the naked failure rate.

We make modest demands regarding data availability: Only the inter-failure times and the failure mechanisms leading to the failure accompanied by the preventive maintenance program are required. This information is available in most modern reliability data banks.

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