Modelling of Dependent Competing Risks by First Passage Times of Wiener Processes

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Abstract

Consider the competing risks situation for a component which may be subject to either a failure or a preventive maintenance action, where the latter will prevent the failure. It is then reasonable to expect a dependence between the failure mechanism and the PM regime. This paper reviews some modelling approaches and introduces a new approach based on modelling of the degradation of a component by means of Wiener processes, with failure corresponding to the first crossing of a certain level, and potential time for maintenance corresponding to the crossing of a certain lower degradation level.

1 Introduction

The idea of competing risks is to model the situation where units are exposed to several risks and fail due to one of them. We observe two random variables for each individual, (Y, δ), where Y is the time to failure of the individual, and δ is the cause of failure.

Our motivation and main application in the present paper is the competing risks situation occurring when a potential component failure at some time X may be avoided by a preventive maintenance (PM) at time Z. The experienced event will in this case be at time $Y = \min(X, Z)$, and it will either be a failure or a PM. It is convenient to define $\delta = I(Z < X)$, where $I(A)$ is the indicator function of the event A. Thus $\delta = 0$ means that the component fails and $\delta = 1$ means that it is preventively maintained.

Note that the observable result is the pair (Y, δ), rather than the underlying times X and Z, which are often the times of interest. For example, knowing the distribution of X would be important as a basis for maintenance optimization. It is well known (Crowder, 2001, Ch. 7), however, that in a competing risks case as described here, the marginal distributions of X and Z are not identifiable from observation of (Y, δ) alone unless specific assumptions are made on the dependence between X and Z. One such assumption is that X and Z are independent (Crowder, 2001, Ch. 7). This assumption is not reasonable in our application, however, since the maintenance crew is likely to have some information regarding the component's state during operation. This insight is used to perform maintenance with the aim of avoiding component failures. We are thus in practice usually faced with a situation of dependent competing risks between X and Z.

Cooke (1993, 1996) introduced the notion of *random signs censoring* which is tailored for such cases. In our notation, random signs censoring can be defined as follows:

Definition 1 *Let (X, Z) be a pair of life variables. Then Z is called a random signs censoring of X if the event $\{Z < X\}$ is stochastically independent of X.*

Thus, random signs censoring means that the event that the failure of the component is preceded by PM, is not influenced by the time X at which the component fails or would have failed without PM. The idea is that the component emits some kind of signal before failure, and that this signal is discovered with a probability which does not depend on the age of the component. Moreover, random signs censoring implies identifiability of the distribution of X, while the distribution of Z is not identifiable in general under these assumptions.

Lindqvist, Støve and Langseth (2006) suggested a model called *the repair alert model* for describing the joint behavior of failure times X and PM-times Z. This model is a special case of random signs censoring obtained by introducing a repair alert function which describes the "alertness" of the maintenance crew as a function of time.

In the present paper we suggest another modelling approach which again leads to a model satisfying the random signs property. The approach is based on modelling of the degradation of a component by means of Wiener processes, with failure corresponding to the first passage time of a certain level. As is well known (Chhikara and Folks, 1989), this implies that the failure time has an inverse Gaussian distribution. The clue is that a PM may be performed when the degradation process reaches a certain level below the failure level. Whitmore (1986) studied a similar case using first passage times of multidimensional Wiener processes to model independent competing risks. Our approach is related to the one of Whitmore, Crowder and Lawless (1998) who considered a marker process which is correlated to the latent failure process. Aalen and Gjessing (2001) gave a review of failure time models based on first passage times of stochastic processes. For an application of inverse Gaussian distributions in accelerated life testing we refer to Doksum and Høyland (1992).

2 Notation

Throughout the paper we assume that (X, Z) is a pair of continuously distributed life variables such that $P(X = Z) = 0$. We let $F_X(t) = P(X \leq t)$ and $F_Z(t) = P(Z \leq t)$ be the cumulative distribution functions of X and Z, respectively. The *subdistribution functions* of X and Z are defined as, respectively, $F_X^*(t) = P(X \leq t, X < Z)$ and $F_Z^*(t) = P(Z \leq t, Z < X)$. Similarly, the *subsurvival functions* are $S_X^*(t) = P(X > t, X < Z)$ and $S_Z^*(t) = P(Z > t, Z < X)$, while the *subdensity functions* are $f_X^*(t) = F_X^{*'}(t) = -S_X^{*'}(t)$ and similarly for Z.

Note that the functions F_X^* and F_Z^* are nondecreasing with $F_X^*(0) = 0$ and $F_Z^*(0) = 0$.

Moreover, we have $F_X^*(\infty) + F_Z^*(\infty) = 1$. Any pair of functions K_1, K_2 satisfying these conditions, will later be referred to as a *subdistribution pair.*

We will also use the notion of *conditional distribution functions*, defined by $\tilde{F}_X(t) = P(X \leq t | X < Z)$ and $\tilde{F}_Z(t) = P(Z \leq t | Z < X)$. Note then that $\tilde{F}_X(t) = F_X^*(t)/F_X^*(\infty)$, $\tilde{F}_Z(t) = F_Z^*(t)/F_Z^*(\infty)$.

3 Random signs and the repair alert model

It was mentioned in the introduction that the marginal distribution of X is identifiable under random signs censoring. This follows directly from Definition 1, since we must have

$$\tilde{F}_X(t) = P(X \leq t | X < Z) = P(X \leq t) = F_X(t).$$

Hence we have the somewhat surprising result that the marginal distribution of X is the same as the distribution of the observed occurrences of X.

The following theorem states that a random signs distribution for (X, Z) exists if and only if the conditional distribution function of X is below that of Z.

Theorem 1 *(Cooke, 1993). Let K_1, K_2 be a subdistribution pair. Then the following are equivalent:*

(i) *There exists a pair (X, Z) of life variables such that Z is a random signs censoring of X, and such that*

$$F_X^*(t) = K_1(t) \text{ for all } t \geq 0, \; F_Z^*(t) = K_2(t)$$

for all $t \geq 0$.

(ii)

$$\frac{K_1(t)}{K_1(\infty)} < \frac{K_2(t)}{K_2(\infty)} \text{ for all } t > 0.$$

The intuitive implication of this result is that the condition $\tilde{F}_X(t) < \tilde{F}_Z(t)$ for all $t > 0$ (corresponding to (ii)), is consistent with Z being a random signs censoring of X.

On the other hand, if $\tilde{F}_X(t) \geq \tilde{F}_Z(t)$ for some t, then there is no joint distribution of (X, Z) for which the random signs requirement holds. For more discussion on random signs censoring and its applications we refer to Cooke (1993, 1996), and Bedford and Cooke (2001, Ch. 9).

The repair alert model is defined as follows.

Definition 2 *(Lindqvist et al. 1996). The pair (X, Z) of life variables satisfies the requirements of the repair alert model provided the following two conditions both hold:*

(i) *The event $\{Z < X\}$ is stochastically independent of X (i.e. Z is a random signs censoring of X).*

(ii) *There exists an increasing function G with $G(0) = 0$ such that for all $x > 0$,*

$$P(Z \leq z | Z < X, X = x) = \frac{G(z)}{G(x)}, \quad 0 < z \leq x \; .$$

The function G is called *the cumulative repair alert function.* Its derivative g is called *the repair alert function.*

The repair alert model is, as already noted, a specialization of random signs censoring, obtained by introducing the repair alert function g. Part (ii) of the definition means that, given a potential failure at time $X = x$, and given that a PM will be performed before that time, the conditional density of the actual time Z of PM is proportional to g. The repair alert function is meant to reflect the reaction of the maintenance crew. Thus $g(t)$ ought to be large at times t for which failures are expected and the alert therefore should be high. Langseth and Lindqvist (2003) simply used $g(t) = \lambda_X(t)$ where λ_X is the hazard rate of the failure time X.

4 The Wiener process and the inverse Gaussian distribution

Instead of viewing a failure as a sudden happening, it can be considered as an ending point of some underlying degradation process. Here we shall assume that the degradation is modelled by a Wiener process as described in Aalen and Gjessing (2001).

Definition 3 *A stochastic process $\{W(t), t \geq 0\}$ is a Wiener process with drift coefficient ν and variance parameter σ^2 if*

1. $W(0) = 0$,
2. $\{W(t), t \geq 0\}$ *has stationary and independent increments,*
3. *for every $t > 0$, $W(t)$ is normally distributed with mean νt and variance $\sigma^2 t$.*

When modelling degradation with a Wiener process it is natural to assume a positive drift coefficient. An illustrative example is given in Figure 1.

A special feature that makes the Wiener process mathematically tractable is that the first passage time to a level $a > 0$ is inverse Gaussian distributed with density

$$f(t; \nu, \sigma, a) = \frac{a}{\sqrt{2\pi}\sigma} t^{-\frac{3}{2}} \exp\left\{-\frac{(a-\nu t)^2}{2t\sigma^2}\right\}, t > 0,$$

(Chhikara and Folks, 1989). From this density it can be seen that the variance parameter σ^2 is appears only in the ratios $\frac{a}{\sigma}$ and $\frac{\nu}{\sigma}$. As noted by Aalen and Gjessing (2001), this means that we can put $\sigma = 1$ without loss of generality. This leads to the density function

$$f(t; \nu, a) = \frac{a}{\sqrt{2\pi}} t^{-\frac{3}{2}} \exp\left\{-\frac{(a-\nu t)^2}{2t}\right\}, t > 0. \tag{1}$$

which will be used later in this paper. We denote this distribution by IG(ν, a), the inverse Gaussian distribution with parameters ν and a. The corresponding survival function is given by

$$S(t; \nu, a) = \Phi\left(\frac{a-\nu t}{\sqrt{t}}\right) - e^{2a\nu}\Phi\left(\frac{-a-\nu t}{\sqrt{t}}\right). \tag{2}$$

where Φ is the standard normal cumulative distribution function.

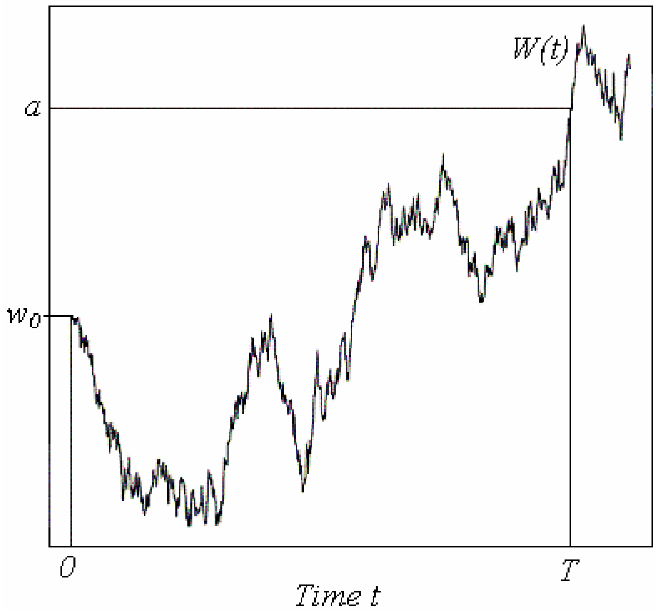

Figure 1: Illustration of a Wiener process with $W(0) = w_0$ and positive drift. The time T is the first passage time to the level $a > w_0$, and therefore inverse Gaussian distributed.

5 Basic Wiener process model for PM

Now assume that the state of the component follows a Wiener process with $W(0) = 0$ and positive drift ν. When the process reaches a level $s > 0$, the item emits a "signal" in the sense of Cooke's random signs censoring. The time this happens is T_s, the first passage time to s, and hence has the inverse Gaussian distribution IG(ν, s) with density $f(t; \nu, s)$ given by (1) and survival function $S(t; \nu, s)$ given by (2). If the signal is detected, the time T_s will be observed, and we thus put $Z = T_s$ and say that a PM is performed at Z. If the signal is not detected, the time Z will not be observed, and the process will go on until it reaches a critical level $c > s$ at time T_c, where the item fails. In this case we observe $X = T_c$ where T_c has the distribution IG(ν, c).

The probability of detecting the signal when the process reaches level s is assumed to be q, $0 < q < 1$, and the event that the signal is detected is assumed to be independent of the Wiener process.

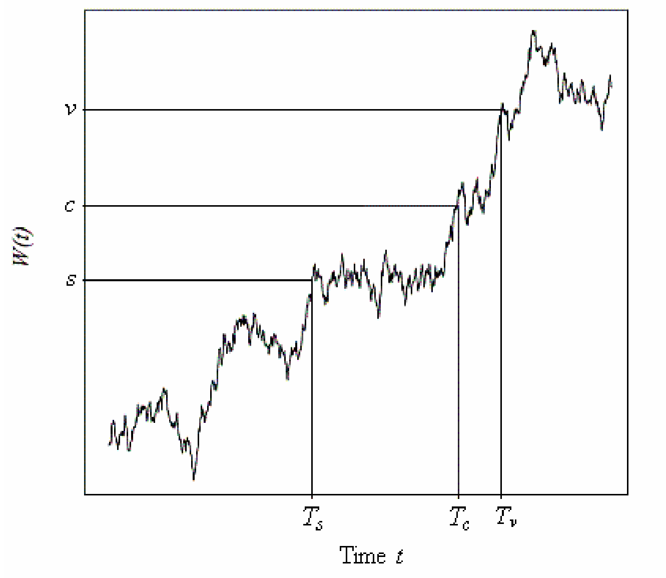

Figure 2: Illustration of a Wiener process with a level s where a signal is emitted. If the signal is detected, then a PM is performed at time $Z = T_s$ and X is not observed. If the signal is not detected, the process goes on to a critical level c, where the corresponding time to failure $X = T_c$ is observed. In this case Z is not observed and can be modelled as any time past T_c, here illustrated by time T_v where $v > c$.

Note that the potential failure time X can be taken to be inverse Gaussian distributed with parameters ν and c whether or not Z is observed. On the other hand, it should be noted that while Z conditionally given $Z < X$ is inverse Gaussian distributed, it is usually not the case that Z itself is inverse Gaussian distributed. This is since Z may be given any value greater than X when the process is not stopped at s.

The situation is described in Figure 2, which shows the path of a Wiener process with positive drift. The levels s and c are indicated together with the corresponding times T_s and T_c. In addition there is indicated a third level $v > c$ with a corresponding time T_v, such that $Z = T_s$ if there is a PM and $Z = T_v$ if a failure is observed. This is only an illustration to show a possible behavior of Z when it is larger than X. The time T_v will, however, never be observed.

We now show that the basic model described above satisfies the requirements of random signs censoring. Recall that Z is a random signs censoring of X if the event $\{Z < X\}$ is independent of X. In the present case, the event $\{Z < X\}$ means that the emitted signal is detected. This happens with probability q, and the event is by assumption independent of the Wiener process and hence independent of X. Thus Z is indeed a random signs censoring of X.

It would also be interesting to see whether the Wiener process model for PM can be represented as a repair alert model. Recall that there are two conditions that X and Z have to satisfy the requirements of a repair alert model. Condition (i) is the random signs condition which is satisfied as already noted. Condition (ii) is equivalent to being able to write the conditional density of T_s given $T_c = t_c$ on the form $\frac{g(t_s)}{G(t_c)}$ for some $g(t) = G'(t)$. Since $f(t_s|t_c) = f(t_s, t_c)/f(t_c)$, a necessary condition for (ii) is that the joint density $f(t_s, t_c)$ factorizes as $h_1(t_s)h_2(t_c)$. Conditional on $T_s = t_s$ we have $T_c = t_s + IG(\nu, c - s)$, so

$$\begin{aligned} f(t_s, t_c) &= f(t_s)f(t_c|t_s) \\ &= \frac{s(c-s)}{2\pi} \left[t_s(t_c - t_s)\right]^{-3/2} \exp\left\{-\frac{(s - \nu t_s)^2}{2t_s}\right. \\ &- \left.\frac{\left[(c-s) - \nu(t_c - t_s)\right]^2}{2(t_c - t_s)}\right\}. \end{aligned}$$

This does not factorize, however, which may be verified empirically by plotting.

The likelihood function

The contribution to the likelihood function when $X = x$ is observed is the subdensity function for X, $f_X^*(x)$. To find $f_X^*(x)$ we first calculate the subsurvival function and then differentiate to get the subdensity function:

$$\begin{aligned} S_X^*(x) &= P(X > x, X < Z) \\ &= P(T_c > x)P(X < Z) \\ &= (1-q)S(x; \nu, c), \end{aligned}$$

which by (1) gives

$$\begin{aligned} f_X^*(x) &= (1-q)f(x;\nu,c) \\ &= (1-q)\frac{c}{\sqrt{2\pi}}x^{-3/2}\exp\left\{-\frac{(c-\nu x)^2}{2x}\right\}. \end{aligned}$$

The contribution when $Z=z$ is observed is the subdensity function $f_Z^*(z)$. The subsurvival function for Z is found by

$$\begin{aligned} S_Z^*(z) &= P(Z>z, Z<X) \\ &= P(Z<X)P(Z>z|Z<X) \\ &= qS(z;\nu,s). \end{aligned}$$

and hence the subdensity function for Z is given by

$$\begin{aligned} f_Z^*(z) &= qf(z;\nu,s) \\ &= q\frac{s}{\sqrt{2\pi}}z^{-3/2}\exp\left\{-\frac{(s-\nu z)^2}{2z}\right\}. \end{aligned}$$

When we observe $x_1,...,x_m$ and $z_1,...,z_n$ the likelihood function is the product of all these observations, i.e.

$$\begin{aligned} L &= \prod_{i=1}^m f_X^*(x_i)\prod_{j=1}^n f_Z^*(z_j) \\ &= (1-q)^m q^n \frac{c^m s^n}{(2\pi)^{\frac{m+n}{2}}}\left(\prod_{i=1}^m x_i\right)^{-3/2}\left(\prod_{j=1}^n z_j\right)^{-3/2} \\ &\quad\times\exp\left\{-\sum_{i=1}^m\frac{(c-\nu x_i)^2}{2x_i}-\sum_{j=1}^n\frac{(s-\nu z_j)^2}{2z_j}\right\}. \end{aligned} \tag{3}$$

Maximum likelihood estimates of the parameters q, c, s and ν can be found by computing the log likelihood function, differentiating and solving the following likelihood equations,

$$mq - n(1-q) = 0, \tag{4}$$

$$\frac{1}{c} - c\frac{1}{m}\sum_{i=1}^m\frac{1}{x_i} + \nu = 0, \tag{5}$$

$$\frac{1}{s} - s\frac{1}{n}\sum_{j=1}^{n}\frac{1}{z_j} + \nu = 0, \tag{6}$$

$$cm + sn = \nu\left(\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} z_j\right). \tag{7}$$

This gives

$$\hat{q} = \frac{n}{n+m},$$

while the estimates of c, s and ν are easily found by numerical methods.

Additional independent censoring

In practice there may be observations which are right censored by some independent source, in which case neither X nor Z is observed, but rather a censoring time τ. Assume therefore that we observe $x_1, ..., x_m$ and $z_1, ..., z_n$ as above, but that for additional components we observe just the censoring times $\tau_1, ..., \tau_r$. The contribution to the likelihood from a censored observation at τ is now

$$\begin{aligned} P(X > \tau, Z > \tau) &= P(X > \tau, X < Z) + P(Z > \tau, Z < X) \\ &= (1-q)S(\tau; \nu, c) + qS(\tau; \nu, s), \end{aligned} \tag{8}$$

The likelihood function L from (3) must therefore be multiplied by the contributions from all censored observations,

$$L_\tau = \prod_{k=1}^{r}\left[(1-q)S_{T_c}(\tau_k) + qS_{T_s}(\tau_k)\right].$$

The resulting log likelihood for data $x_1, ..., x_m$, $z_1, ..., z_n$, $\tau_1, ..., \tau_r$ now becomes

$$
\begin{aligned}
l &= m \ln(1-q) + n \ln q - \frac{m+n}{2} \ln 2\pi + m \ln c \\
&+ n \ln s - \frac{3}{2} \sum_{i=1}^{m} \ln x_i - \frac{3}{2} \sum_{i=1}^{n} \ln z_i \\
&- \sum_{i=1}^{m} \frac{(c-\nu x_i)^2}{2x_i} - \sum_{j=1}^{n} \frac{(s-\nu z_j)^2}{2z_j} \\
&+ \sum_{k=1}^{r} \ln \left[(1-q) \left(\Phi\left(\frac{c-\nu\tau_k}{\sqrt{\tau_k}}\right) \right.\right. \\
&- \left.\left. e^{2c\nu} \Phi\left(\frac{-c-\nu\tau_k}{\sqrt{\tau_k}}\right) \right)\right. \\
&+ \left. q \left(\Phi\left(\frac{s-\nu\tau_k}{\sqrt{\tau_k}}\right) - e^{2s\nu} \Phi\left(\frac{-s-\nu\tau_k}{\sqrt{\tau_k}}\right) \right) \right],
\end{aligned}
$$

which by maximization gives the maximum likelihood estimates of the parameters. The likelihood equations (4)-(7) are of course no longer valid, and numerical methods are now needed even for the estimation of q.

6 Wiener process model with random level S

In the basic model presented in the previous section, the component is assumed to emit a signal at a fixed level s, which is the same for all components. A way to extend this model is to let the level s be a random variable, S. This means in effect that the components are heterogeneous with respect to the signal level. For example, such a heterogeneity can be explained by variations in maintenance policies. Note in the following that the potential time Z to PM now is given by T_S.

Now make the assumption that the random level S is independent of the Wiener process. Since X as before is assumed to equal T_c for a fixed critical level c, it is clear that the requirements of random signs censoring still hold. In fact, the event $Z < X$ now corresponds to $S < c$ which is clearly independent of T_c.

It should be clear that the basic model of the previous section is actually a special case of the model with random level S. In fact, for the basic model specified by a fixed signal

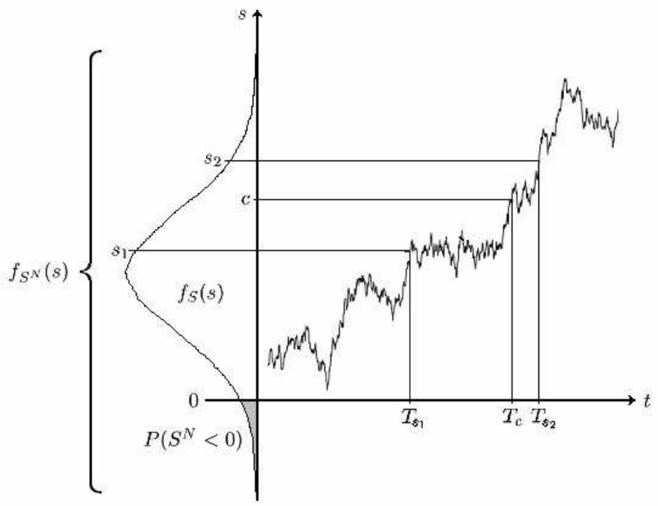

Figure 3: Illustration of a Wiener process model with a random level S defined as the left truncation at 0 of a normal random variable S^N. The critical level c is fixed. Two realizations, s_1 and s_2, of S are illustrated. In the case $S = s_1$ we have that $Z = T_{s_1}$ is observed, while in the case $S = s_2$ it is seen that $X = T_c$ is observed.

level s and critical level c we may define a random variable S with two possible values, s and v, say, such that $P(S = s) = q$ and $P(S = v) = 1 - q$ and where $v > c$ (see Figure 2).

For the general case, let $F_S(s)$ and $f_S(s)$ be the distribution function and density function of S, respectively. In order to make sense under an assumed positive trend parameter ν we need to have $P(S > 0) = 1$ and $P(S < c) > 0$. Skogsrud (2005) suggested a truncated normal distribution and a uniform distribution as useful distributions of S. Figure 3 illustrates the normal model.

The subdistribution function of X is now given by

$$
\begin{aligned}
S_X^*(x) &= P(X > x, X < Z) \\
&= P(T_c > x, S > c) \\
&= P(T_c > x)P(S > c) \\
&= (1 - F_S(c))S(x; \nu, c).
\end{aligned}
$$

Note that $F_S(c) = P(S \le c)$ corresponds to $q = P(Z < X)$ from the basic Wiener model.

The subdensity of X is found by differentiating S^*_X and is thus

$$f^*_X(x) = (1 - F_S(c)) f(x; \nu, c).$$

The subsurvival and subdensity functions of Z are slightly more complicated than in the basic Wiener case since T_S and S now are stochastically dependent. More precisely we obtain

$$\begin{aligned} S^*_Z(z) &= P(Z > z, Z < X) \\ &= P(T_S > z, S < c) \\ &= \int_0^c P(T_S > z, s \leq S \leq s + \mathrm{d}s) \\ &= \int_0^c P(T_s > z, s \leq S \leq s + \mathrm{d}s) \\ &= \int_0^c P(T_s > z) P(s \leq S \leq s + \mathrm{d}s) \\ &= \int_0^c S(z; \nu, s) f_S(s) \mathrm{d}s. \end{aligned}$$

and hence by differentiation,

$$\begin{aligned} f^*_Z(z) &= \int_0^c f(z; \nu, s) f_S(s) \mathrm{d}s \\ &= \int_0^c \frac{s}{\sqrt{2\pi}} z^{-\frac{3}{2}} \exp\left\{ -\frac{(s - \nu z)^2}{2z} \right\} f_S(s) \mathrm{d}s. \end{aligned} \tag{9}$$

In order to write down the full likelihood function in the case of censored observations, we also need the contribution of an observation censored at τ, which is

$$\begin{aligned} P(X > \tau, Z > \tau) &= P(T_c > \tau, T_S > \tau) \\ &= \int_0^\infty P(T_c > \tau, T_S > \tau, s \leq S \leq s + \mathrm{d}s) \\ &= \int_0^c P(T_c > \tau, T_s > \tau) f_S(s) \mathrm{d}s + \int_c^\infty P(T_c > \tau, T_s > \tau) f_S(s) \mathrm{d}s \\ &= \int_0^c P(T_s > \tau) f_S(s) \mathrm{d}s + P(T_c > \tau) \int_c^\infty f_S(s) \mathrm{d}s \\ &= \int_0^c S(\tau; \nu, s) f_S(s) \mathrm{d}s + S(\tau; \nu, c)(1 - F_S(c)). \end{aligned} \tag{10}$$

Note that this generalizes the expression (8) for the basic model. In fact, the probability $F_S(c)$ is then the same as q, while the integral $\int_0^c S(\tau;\nu,s)f_S(s)\mathrm{d}s$ reduces to $qS(\tau;\nu,s)$.

This finally leads to the following likelihood function for observations $x_1,...,x_m$, $z_1,...,z_n$ and $\tau_1,...,\tau_r$ for the model with random level S and independent censoring:

$$\begin{aligned} L &= \prod_{i=1}^{m} f_X^*(x_i)\prod_{j=1}^{n} f_Z^*(z_j)\prod_{k=1}^{r} P(X>\tau_k, Z>\tau_k) \\ &= \prod_{i=1}^{m}(1-F_S(c))\frac{c}{\sqrt{2\pi}}x_i^{-\frac{3}{2}}\exp\left\{-\frac{(c-\nu x_i)^2}{2x_i}\right\}\prod_{j=1}^{n} f_Z^*(z_j) \\ &\quad\times\prod_{k=1}^{r} P(X>\tau_k, Z>\tau_k), \end{aligned}$$

where the functions $f_Z^*(z)$ and $P(X>\tau, Z>\tau)$ are given in (9) and (10), respectively. Note that both these functions will depend on the distribution of S. Skogsrud (2005) obtained explicit expressions for the likelihood when S is, respectively, truncated normal and uniform on an interval $[0,A]$.

7 Example: VHF-data

Mendenhall and Hader (1958) presented data of times to failure for ARC-1 VHF communication transmitter-receivers of a single commercial airline. The times to failure were actually times to removal of units that were assumed to be failed. After the removal, the units were sent to maintenance and it turned out that some of the units were not failed after all. Time to failure for the confirmed failures will here be represented by X, while time to removal of the units with unconfirmed failures is represented by Z. In addition the observations were censored at time $\tau = 630$ hours because the airline removed every unit which had been operated for that long. (This is a type I censoring). There are $m = 218$ observations of X, $n = 107$ observations of Z and $r = 44$ censored observations. In the following these data will be referred to as the VHF-data.

Nonparametric estimates of $\tilde{S}_X(t) = 1-\tilde{F}_X(t)$ and $\tilde{S}_X(t) = 1-\tilde{F}_X(t)$ are given in Figure 4, computed by means of formulas in Lawless (2003, Ch. 9.2). These indicate that the condition for random signs censoring, $\tilde{F}_X(t) < \tilde{F}_Z(t)$, holds for these data at least for $t > 100$, while the situation is not that clear for $t < 100$. Still we assume that a random signs censoring model can be applied to the data.

Maximum likelihood estimates of the parameters ν, c, s and q for the basic Wiener process model are displayed in Table 1, which also gives approximate 95% confidence intervals for each parameter.

Table 1: Table of maximum likelihood estimates of the parameters ν, c, s and q for the VHF-data in the basic Wiener process model with censoring. In addition the standard deviation and 95% confidence intervals are included using standard normal theory.

Parameter	Estimate	St deviation	Lower bound	Upper bound
ν	0.03412	0.003838	0.02737	0.04254
c	12.64	0.5780	11.56	13.83
s	10.64	0.6762	9.392	12.05
q	0.3230	0.02592	0.2760	0.3780

When we assume the normal random level model, the parameters are estimated as shown in Table 2. The estimated drift ν is close to the one estimated in the basic model, but slightly higher. This is apparently compensated in the estimated critical level c, which also lies a bit above the estimate from the previous model.

Table 2: Table of maximum likelihood estimates of the parameters ν, c, μ_S and σ_S for the VHF-data in the Wiener process model with random S given by a truncated normal distribution with mean μ_S and variancve σ_S^2. In addition the standard deviation and 95% confidence intervals are included using standard normal theory.

Parameter	Estimate	Standard deviation	Lower bound	Upper bound
ν	0.03629	0.003927	0.02935	0.04486
c	13.26	0.5744	12.18	14.43
μ_S	14.87	0.8605	13.28	16.66
σ_S	3.489	0.7143	2.336	5.212

We can estimate the probability of observing Z in the normal model by

$$\hat{q} = \hat{P}(Z < X) = \hat{P}(S < c) = \Phi\left(\frac{\hat{c} - \hat{\mu}_S}{\hat{\sigma}_S}\right) = \Phi(-0.4615) = 0.3222.$$

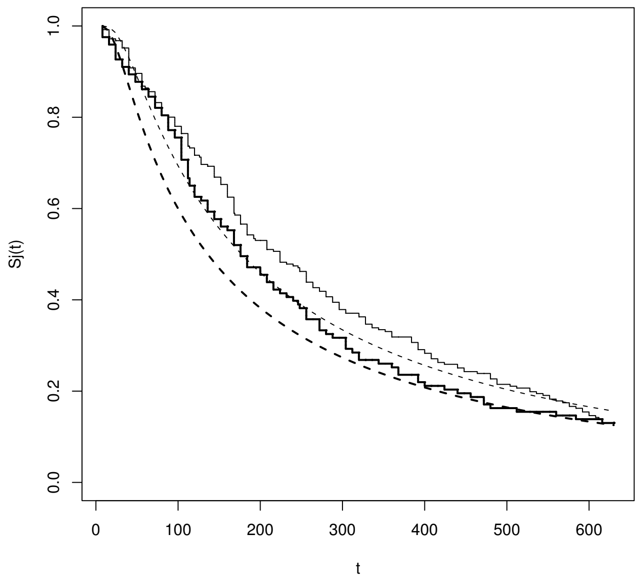

Figure 4: Parametric estimated conditional survival functions $\hat{\hat{S}}_X(t)$ (thin dashed line) and $\hat{\hat{S}}_Z(t)$ (thick dashed line) for the basic models in the VHF-data. Plotted with the non-parametric estimates of $\tilde{S}_X(t)$ (thin line) and $\tilde{S}_Z(t)$ (thick line).

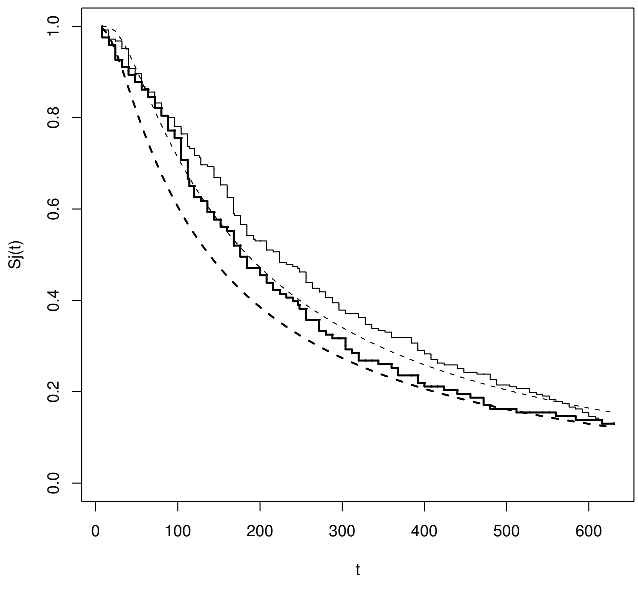

Figure 5: Parametric estimated conditional survival functions $\hat{\hat{S}}_X(t)$ (thin dashed line) and $\hat{\hat{S}}_Z(t)$ (thick dashed line) for the normal random level model with the VHF-data. Plotted with the non-parametric estimates of $\tilde{S}_X(t)$ (thin line) and $\tilde{S}_Z(t)$ (thick line).

which is very close to the result of the basic model.

The parametric estimates of $\tilde{S}_X(t)$ and $\tilde{S}_Z(t)$ for the two models are plotted in Figure 4 and Figure 5, respectively. The fits appear to be not much different for the two models, and both are somewhat unsatisfactory, except for t large ($t > 400$). A slight difference in favor of the normal random level model is indicated by looking at the maximum log likelihood values of the two models. These are -2406.3 for the basic model and -2401.9 for the normal random level model. The conclusion from this higher number for the normal random model is not clear, however, since the models are not nested but have the same number of parameters.

The reason for the unsatisfactory fit, as discussed above, is presumably that the inverse Gaussian distribution is not a good model for the failure times X. For comparison, we plotted also, in Figure 6, the corresponding parametrically estimated curves for the repair alert model presented in Section 3. Analysis of the VHF-data by means of an exponential repair alert model is performed in Lindqvist et al. (2006), with X being exponentially distributed with hazard rate λ and the cumulative repair alert function given by $G(t) = t^\beta$. Maximum likelihood estimates of λ and β were calculated to be $\hat{\lambda} = 3.10 \cdot 10^{-3}$ and $\hat{\beta} = 4.44$, while q was estimated as 0.318. The fit to the data seems to be much better for this model as seen from Figure 6.

8 Concluding remarks

The non-identifiability of marginal distributions is well known in competing risks situations and the issue is well described, for example, in the book by Crowder (2001). The problem is apparent in reliability analyses, since the estimation of marginal failure distributions is of primary importance there. Random signs censoring is an interesting option in such studies, but like all other approaches this is again based on non-testable assumptions.

Instead of the Wiener process models studied in the present paper, we may use other types of processes, for example gamma processes. An intuitive advantage of the latter

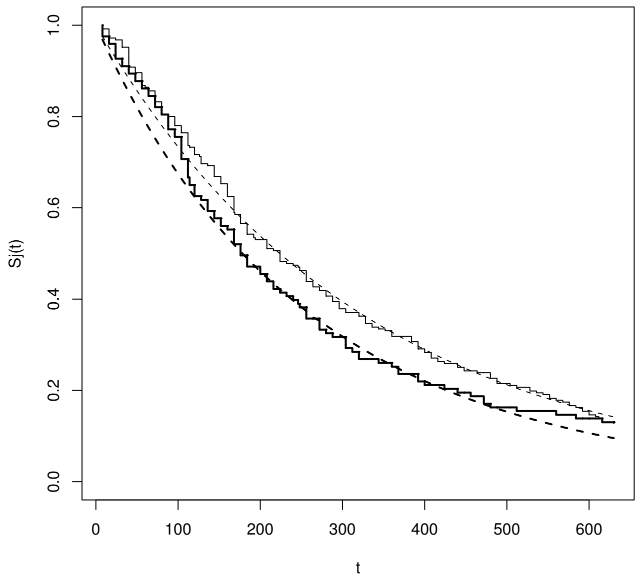

Figure 6: Parametric estimated conditional survival functions $\hat{\tilde{S}}_X(t)$ (thin dashed line) and $\hat{\tilde{S}}_X(t)$ (thick dashed line) for the repair alert model with $f_X(x) = \lambda e^{-\lambda x}$ and $G(t) = t^{\beta}$ applied on the VHF-data (Lindqvist and Langseth, 2005). Plotted with the non-parametric estimates of $\tilde{S}_X(t)$ (thin line) and $\tilde{S}_Z(t)$ (thick line)

processes is that they are strictly increasing, which seems more reasonable for a degradation process.

Skogsrud (2005) considered the extension of the Wiener process model presented here, obtained by letting the level s of PM be a random variable. The models can also be extended to include the possibility of covariates. This can be done in a way similar to the one described by Aalen and Gjessing (2001). For example, we could let the level c of failure depend on the covariates. Also, we could let covariates influence the drift parameter ν.

References

Aalen, O. O. and Gjessing, H. (2001) Understanding the shape of the hazard rate: a process point of view. Statistical Science, 16, 1–22.

Bedford, T. and Cooke, R. M. (2001) Probabilistic risk analysis: Foundations and methods. Cambridge University Press, Cambridge.

Chhikara, R. S. and Folks, J. L. (1989). The Inverse Gaussian Distribution: Theory, Methodology and Applications. Marcel Dekker, New York.

Cooke, R. M. (1993) The total time on test statistics and age-dependent censoring. Statistics and Probability Letters, 18, 307–312.

Cooke, R. M. (1996) The design of reliability databases, Part I and II. Reliability Engineering and System Safety, 51, 137–146 and 209–223.

Crowder, M. J. (2001) Classical Competing Risks. Chapman and Hall/CRC, Boca Raton.

Doksum, K. A. and Høyland, A. (1992). Models for variable stress accelerated life testing experiments based on Wiener processes and the inverse gaussian distribution. Technometrics, 34, 74-82.

Langseth, H. and Lindqvist, B. H. (2003) A maintenance model for components exposed to several failure mechanisms and imperfect repair. In Mathematical and Statistical

Methods in Reliability (Eds. B. H. Lindqvist and K. A. Doksum). World Scientific Publishing, Singapore, pp. 415-430.

Lawless, J. F. (2003) Statistical models and methods for lifetime data, 2nd ed. Wiley-Interscience, Hoboken, NJ.

Lindqvist, B. H. and Langseth, H. (2005) Statistical modelling and inference for component failure times under preventive maintenance and independent censoring. In Modern Statistical and Mathematical Methods in Reliability (Eds. A. Wilson, N. Limnios, S. Keller-McNulty, and Y. Armijo). World Scientific Publishing, Singapore, pp. 323–337.

Lindqvist, B. H., Støve, B. and Langseth, H. (2006) Modelling of dependence between critical failure and preventive maintenance: The repair alert model. Journal of Statistical Planning and Inference, 136, 1701–1717.

Mendenhall, W. and Hader, R. J. (1958) Estimation of parameters of mixed exponentially distributed failure time distributions from censored life test data. Biometrika, 45, 504–520.

Skogsrud, G. (2005) Statistical modelling of dependent competing risks by means of Wiener processes and the inverse Gaussian distribution. Master Thesis, Department of Mathematical Sciences, Norwegian University of Science and Technology.

Whitmore, G. A. (1986). First-passage-time models for duration data: regression structures and competing risks. The Statistician, 35, 207–219.

Whitmore, G. A., Crowder, M. J. and Lawless, J. F. (1998). Failure inference from a marker process based on a bivariate Wiener model. Lifetime Data Anal. 4 229-251.