Modelling of Dependent Competing Risks by First Passage Times of Stochastic Processes

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***Abstract*— Consider the competing risks situation for a component which may be subject to either a failure or a preventive maintenance action, where the latter will prevent the failure. It is then reasonable to expect a dependence between the failure mechanism and the PM regime. This paper reviews some modelling approaches and introduces a new approach based on modelling of the degradation of a component by means of Wiener processes, with failure corresponding to the first crossing of a certain level, and potential time for maintenance corresponding to the crossing of a certain lower degradation level.**

I. Introduction

The idea of competing risks is to model the situation where units are exposed to several risks and fail due to one of them. We observe two random variables for each individual, (Y, δ), where Y is the time to failure of the individual, and δ is the cause of failure.

Our motivation and main application in the present paper is the competing risks situation occurring when a potential component failure at some time X may be avoided by a preventive maintenance (PM) at time Z. The experienced event will in this case be at time $Y = \min(X, Z)$, and it will either be a failure or a PM. It is convenient to define $\delta = I(Z < X)$, where $I(A)$ is the indicator function of the event A. Thus $\delta = 0$ means that the component fails and $\delta = 1$ means that it is preventively maintained.

Note that the observable result is the pair (Y, δ), rather than the underlying times X and Z, which are often the times of interest. For example, knowing the distribution of X would be important as a basis for maintenance optimization. It is well known [5], however, that in a competing risks case as described here, the marginal distributions of X and Z are not identifiable from observation of (Y, δ) alone unless specific assumptions are made on the dependence between X and Z. One such assumption is that X and Z are independent ([5]). This assumption is not reasonable in our application, however, since the maintenance crew is likely to have some information regarding the component's state during operation. This insight is used to perform maintenance with the aim of avoiding component failures. We are thus in practice usually faced with a situation of dependent competing risks between X and Z.

Cooke [3], [4] introduced the notion of *random signs censoring* which is tailored for such cases. In our notation, random signs censoring can be defined as follows:

Definition 1: Let (X, Z) be a pair of life variables. Then Z is called a random signs censoring of X if the event $\{Z < X\}$ is stochastically independent of X.

Thus, random signs censoring means that the event that the failure of the component is preceded by PM, is not influenced by the time X at which the component fails or would have failed without PM. The idea is that the component emits some kind of signal before failure, and that this signal is discovered with a probability which does not depend on the age of the component. Moreover, random signs censoring implies identifiability of the distribution of X, while the distribution of Z is not identifiable in general under these assumptions.

Lindqvist, Støve and Langseth [9] suggested a model called *the repair alert model* for describing the joint behavior of failure times X and PM-times Z. This model is a special case of random signs censoring obtained by introducing a repair alert function which describes the "alertness" of the maintenance crew as a function of time.

In the present paper we suggest another modelling approach which again leads to a model satisfying the random signs property. The approach is based on modelling of the degradation of a component by means of stochastic processes, with failure corresponding to the first passage time of a certain level. The clue is that a PM may be performed when the degradation process reaches a certain level below the failure level.

II. Notation

Throughout the paper we assume that (X, Z) is a pair of continuously distributed life variables. We let $F_X(t) = P(X \leq t)$ and $F_Z(t) = P(Z \leq t)$ be the cumulative distribution functions of X and Z, respectively. The *subdistribution functions* of X and Z are defined as, respectively, $F_X^*(t) = P(X \leq t, X < Z)$ and $F_Z^*(t) = P(Z \leq t, Z < X)$. Similarly, the *subsurvival functions* are $S_X^*(t) = P(X > t, X < Z)$ and $S_Z^*(t) = P(Z > t, Z < X)$, while the *subdensity functions* are $f_X^*(t) = F_X^{*'}(t) = -S_X^{*'}(t)$ and similarly for Z.

Note that the functions F_X^* and F_Z^* are nondecreasing with $F_X^*(0) = 0$ and $F_Z^*(0) = 0$. Moreover, we have $F_X^*(\infty) + F_Z^*(\infty) = 1$. Any pair of functions K_1, K_2 satisfying these conditions, will later be referred to as a *subdistribution pair*.

We will also use the notion of *conditional distribution functions*, defined by $\tilde{F}_X(t) = P(X \leq t|X < Z)$ and $\tilde{F}_Z(t) = P(Z \leq t|Z < X)$. Note then that $\tilde{F}_X(t) = F_X^*(t)/F_X^*(\infty)$, $\tilde{F}_Z(t) = F_Z^*(t)/F_Z^*(\infty)$.

III. Random signs and the repair alert model

It was mentioned in the introduction that the marginal distribution of X is identifiable under random signs censoring. This follows directly from Definition 1, since we must have

$$\tilde{F}_X(t) = P(X \leq t|X < Z) = P(X \leq t) = F_X(t).$$

Hence we have the somewhat surprising result that the marginal distribution of X is in fact the same as the distribution of the observed occurrences of X.

The following theorem states that a random signs distribution for (X, Z) exists if and only if the conditional distribution function of X is below that of Z.

Theorem 1: (Cooke [3].) Let K_1, K_2 be a subdistribution pair. Then the following are equivalent:

(i) There exists a pair (X, Z) of life variables such that Z is a random signs censoring of X, and such that

$$F_X^*(t) = K_1(t) \text{ for all } t \geq 0,\ F_Z^*(t) = K_2(t)$$

for all $t \geq 0$.

(ii)

$$\frac{K_1(t)}{K_1(\infty)} < \frac{K_2(t)}{K_2(\infty)} \text{ for all } t > 0.$$

The intuitive implication of this result is that the condition $\tilde{F}_X(t) < \tilde{F}_Z(t)$ for all $t > 0$ (corresponding to (ii)), is consistent with Z being a random signs censoring of X.

On the other hand, if $\tilde{F}_X(t) \geq \tilde{F}_Z(t)$ for some t, then there is no joint distribution of (X, Z) for which the random signs requirement holds. For more discussion on random signs censoring and its applications we refer to Cooke [3], [4], and Bedford and Cooke [2, Ch. 9].

The repair alert model is defined as follows.

Definition 2: (Lindqvist et al. [9]). The pair (X, Z) of life variables satisfies the requirements of the repair alert model provided the following two conditions both hold:

(i) The event $\{Z < X\}$ is stochastically independent of X (i.e. Z is a random signs censoring of X).

(ii) There exists an increasing function G with $G(0) = 0$ such that for all $x > 0$,

$$P(Z \leq z|Z < X, X = x) = \frac{G(z)}{G(x)},\ \ 0 < z \leq x\ .$$

The function G is called *the cumulative repair alert function*. Its derivative g is called *the repair alert function*.

The repair alert model is, as already noted, a specialization of random signs censoring, obtained by introducing the repair alert function g. Part (ii) of the definition means that, given a potential failure at time $X = x$, and given that a PM will be performed before that time, the conditional density of the actual time Z of PM is proportional to g. The repair alert function is meant to reflect the reaction of the maintenance crew. Thus $g(t)$ ought to be large at times t for which failures are expected and the alert therefore should be high. Langseth and Lindqvist [6] simply used $g(t) = \lambda_X(t)$ where λ_X is the hazard rate of the failure time X.

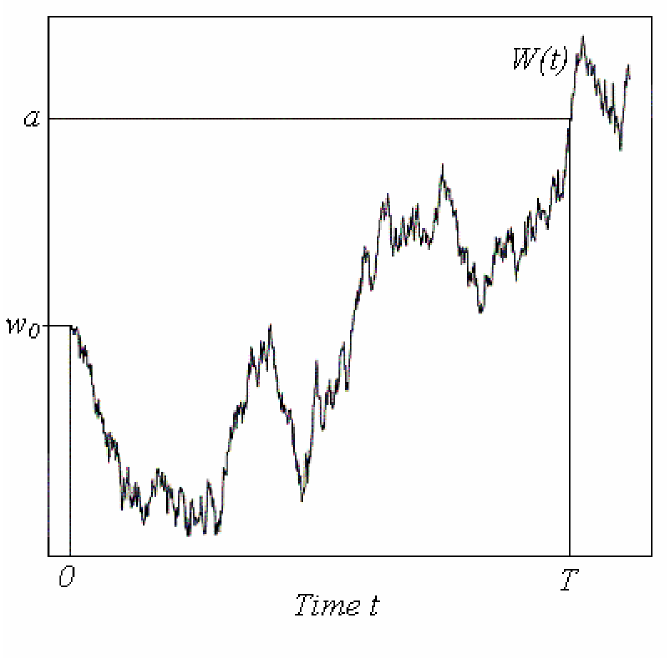

Fig. 1. Illustration of a Wiener process with $W(0) = w_0$ and positive drift. The time T is the first passage time to the level $a > w_0$, and therefore inverse Gaussian distributed.

IV. The Wiener process and the inverse Gaussian distribution

Instead of viewing an event observed in time as a sudden happening, it can be considered as an ending point of some underlying process, in reliability usually called a degradation process. Here we shall assume that the degradation is modelled by a Wiener process as described in Aalen and Gjessing [1].

Definition 3: A stochastic process $\{W(t), t \geq 0\}$ is a *Wiener process* with drift coefficient ν and variance parameter σ^2 if

1) $W(0) = 0$,
2) $\{W(t), t \geq 0\}$ has stationary and independent increments,
3) for every $t > 0$, $W(t)$ is normally distributed with mean νt and variance $\sigma^2 t$.

When modelling degradation with a Wiener process it is natural to assume a positive drift coefficient. An illustrative example is given in Figure 1.

A special feature that makes the Wiener process mathematically tractable is that the first time the process hits a given level is inverse Gaussian distributed. More precisely, the first passage time to a level $a > 0$ is inverse Gaussian distributed with density

$$f(t; \nu, \sigma, a) = \frac{a}{\sqrt{2\pi}\sigma} t^{-\frac{3}{2}} \exp\left\{-\frac{(a - \nu t)^2}{2t\sigma^2}\right\}, t > 0.$$

From this density it can be seen that the variance parameter σ^2 is appearing only in the ratios $\frac{a}{\sigma}$ and $\frac{\nu}{\sigma}$. As noted by Aalen and Gjessing [1], this means that we can put $\sigma = 1$ without loss of generality, which leads to the following density function

$$f(t;\nu,a) = \frac{a}{\sqrt{2\pi}} t^{-\frac{3}{2}} \exp\left\{-\frac{(a-\nu t)^2}{2t}\right\}, t > 0. \quad (1)$$

This is the density function that will be used later in this paper and we shall denote the distribution by IG(ν, a) and call it the inverse Gaussian distribution with parameters ν and a. The corresponding survival function is given by

$$S(t;\nu,a) = \Phi\left(\frac{a-\nu t}{\sqrt{t}}\right) - e^{2a\nu}\Phi\left(\frac{-a-\nu t}{\sqrt{t}}\right). \quad (2)$$

where Φ is the standard normal cumulative distribution function.

V. Wiener process model for PM

Now assume that the state of the component follows a Wiener process with positive drift ν. When the process reaches a level $s > 0$, the item emits a "signal" in the sense of Cooke's random signs censoring. The time this happens is T_s, the first passage time to s, which has an inverse Gaussian distribution with parameters ν and s. If the signal is detected, the time T_s will be observed, and we then put $Z = T_s$ and say that a PM is performed at Z.

If the signal is not detected, the time Z will not be observed, and the process will go on until it reaches the critical level $c > s$ at time T_c, where the item fails. In this case we observe $X = T_c$ where T_c has an inverse Gaussian distribution with parameters ν and c.

The probability of detecting the signal when reaching level s is assumed to be q, and the "draw" made to decide whether to observe Z and perform a PM, is assumed independent of the Wiener process.

Note that the potential failure time X can be taken to be inverse Gaussian distributed with parameters ν and c whether or not Z is observed. On the other hand, regarding Z, it is Z conditionally given $Z < X$ which is inverse Gaussian distributed, not necessarily Z itself. This is since Z may have any value $> X$ when the process is not stopped at s.

The situation is described in Figure 2, which shows the path of a Wiener process with positive drift. The levels s and c are indicated, together with the corresponding times T_s and T_c. In addition there is indicated a third level $v > c$ with a corresponding time T_v, such that $Z = T_s$ if there is a PM and $Z = T_v$ if a failure is observed. This is only an illustration to show a possible behavior of Z when it is larger than X. However, the time T_v will never be observed.

We now show that the basic model satisfies the requirements of random signs censoring. Recall that Z is a random signs censoring of X if the event $\{Z < X\}$ is independent of X. In the present case, the event $\{Z < X\}$ means that the emitted signal is detected. This happens with probability q, and the event is by assumption independent of the Wiener process and

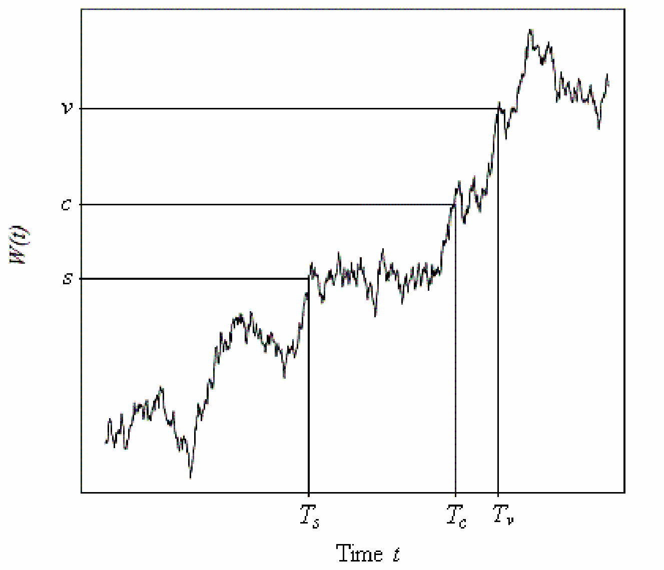

Fig. 2. Illustration of a Wiener process with a level s where a signal is emitted and corresponding time to PM, $Z = T_s$. If the signal is not detected, the process goes on to a critical level c, where the corresponding time to failure $X = T_c$ is observed. In this case Z can be any time past T_c, here illustrated by time T_v.

hence independent of X. Thus Z is indeed a random signs censoring of X.

It would also be interesting to see if the Wiener process model for PM can be represented as a repair alert model. Recall that there are two conditions that X and Z have to satisfy to be modelled by a repair alert model. Condition (i) is the random signs condition which is satisfied as already noted. Condition (ii) is seen to be equivalent to being able to write the conditional density of T_s given $T_c = t_c$ on the form $\frac{g(t_s)}{G(t_c)}$ for some $g(t) = G'(t)$. Since $f(t_s|t_c) = f(t_s,t_c)/f(t_c)$, a necessary condition for (ii) is that the joint density $f(t_s,t_c)$ factorizes as $h_1(t_s)h_2(t_c)$. Conditional on $T_s = t_s$ we have $T_c = t_s + IG(\nu, c-s)$, so

$$\begin{aligned} f(t_s,t_c) &= f(t_s)f(t_c|t_s) \\ &= \frac{s(c-s)}{2\pi}\left[t_s(t_c-t_s)\right]^{-3/2}\exp\left\{-\frac{(s-\nu t_s)^2}{2t_s}\right. \\ &- \left.\frac{\left[(c-s)-\nu(t_c-t_s)\right]^2}{2(t_c-t_s)}\right\}. \end{aligned}$$

This does not factorize, however, which for example can be verified empirically by plotting.

The likelihood function

The contribution to the likelihood function when $X = x$ is observed is the subdensity function for X, $f_X^*(x)$. To find $f_X^*(x)$ we first calculate the subsurvival function and then differentiate to get the subdensity function:

$$\begin{aligned} S_X^*(x) &= P(X > x, X < Z) \\ &= P(T_c > x)P(X < Z) \\ &= (1-q)S_{T_c}(x), \end{aligned}$$

which by (1) gives the following subdensity function for T_c:

$$\begin{aligned} f_X^*(x) &= (1-q)f_{T_c}(x) \\ &= (1-q)\frac{c}{\sqrt{2\pi}}x^{-3/2}\exp\left\{-\frac{(c-\nu x)^2}{2x}\right\}. \end{aligned}$$

The contribution when $Z = z$ is observed is the subdensity function $f_Z^*(z)$. The subsurvival function for Z is found by

$$\begin{aligned} S_Z^*(z) &= P(Z > z, Z < X) \\ &= P(Z < X)P(Z > z | Z < X) \\ &= qS_{T_s}(z). \end{aligned}$$

The subdensity function for Z is then given by

$$\begin{aligned} f_Z^*(z) &= qf_{T_s}(z) \\ &= q\frac{s}{\sqrt{2\pi}}z^{-3/2}\exp\left\{-\frac{(s-\nu z)^2}{2z}\right\}. \end{aligned}$$

When we observe $x_1, ..., x_m$ and $z_1, ..., z_n$ the likelihood function is the product corresponding to of all these observations, i.e.

$$\begin{aligned} L &= \prod_{i=1}^m f_X^*(x_i)\prod_{j=1}^n f_Z^*(z_j) \\ &= (1-q)^m q^n \frac{c^m s^n}{(2\pi)^{\frac{m+n}{2}}}\left(\prod_{i=1}^m x_i\right)^{-3/2}\left(\prod_{j=1}^n z_j\right)^{-3/2} \\ &\quad \times \exp\left\{-\sum_{i=1}^m \frac{(c-\nu x_i)^2}{2x_i} - \sum_{j=1}^n \frac{(s-\nu z_j)^2}{2z_j}\right\}. \quad (3) \end{aligned}$$

Maximum likelihood estimates of the parameters q, c, s and ν can now be found by computing the log likelihood function, differentiating and solving the following likelihood equations,

$$mq - n(1-q) = 0, \quad (4)$$

$$\frac{1}{c} - c\frac{1}{m}\sum_{i=1}^m \frac{1}{x_i} + \nu = 0, \quad (5)$$

$$\frac{1}{s} - s\frac{1}{n}\sum_{j=1}^n \frac{1}{z_j} + \nu = 0, \quad (6)$$

$$cm + sn = \nu\left(\sum_{i=1}^m x_i + \sum_{j=1}^n z_j\right). \quad (7)$$

This gives

$$\hat{q} = \frac{n}{n+m},$$

while the estimates of c, s and ν are easily found by numerical methods.

Additional independent censoring

Assume that we observe $x_1, ..., x_m$ and $z_1, ..., z_n$ as above, but that for additional components we observe only censoring times $\tau_1, ..., \tau_r$ for which it is only known that both Z and X are above the corresponding τ_k. The contribution to the likelihood from a censored observation at τ is now

$$\begin{aligned} P(X > \tau, Z > \tau) &= P(X > \tau, X < Z) \\ &+ P(Z > \tau, Z < X) \\ &= (1-q)S_{T_c}(\tau) + qS_{T_s}(\tau), \end{aligned}$$

where the functions $S_{T_c}(\tau)$ and $S_{T_s}(\tau)$ are the survival function from the inverse Gaussian distribution and can be calculated by using (2). The likelihood function L from (3) should therefore be multiplied by the contributions from all censored observations,

$$L_\tau = \prod_{k=1}^r \left[(1-q)S_{T_c}(\tau_k) + qS_{T_s}(\tau_k)\right].$$

The resulting log likelihood for data $x_1, ..., x_m$, $z_1, ..., z_n$, $\tau_1, ..., \tau_r$ now becomes

$$\begin{aligned} l &= m\ln(1-q) + n\ln q - \frac{m+n}{2}\ln 2\pi + m\ln c \\ &+ n\ln s - \frac{3}{2}\sum_{i=1}^m \ln x_i - \frac{3}{2}\sum_{i=1}^n \ln z_i \\ &- \sum_{i=1}^m \frac{(c-\nu x_i)^2}{2x_i} - \sum_{j=1}^n \frac{(s-\nu z_j)^2}{2z_j} \\ &+ \sum_{k=1}^r \ln\Bigg[(1-q)\Bigg(\Phi\left(\frac{c-\nu\tau_k}{\sqrt{\tau_k}}\right) \\ &- e^{2c\nu}\Phi\left(\frac{-c-\nu\tau_k}{\sqrt{\tau_k}}\right)\Bigg) \\ &+ q\left(\Phi\left(\frac{s-\nu\tau_k}{\sqrt{\tau_k}}\right) - e^{2s\nu}\Phi\left(\frac{-s-\nu\tau_k}{\sqrt{\tau_k}}\right)\right)\Bigg]. \end{aligned}$$

This of course makes the likelihood equations (4)-(7) no longer valid and they should be replaced by more complicated equations which need numerical methods even for the estimation of q.

VI. Example: VHF-data

Mendenhall and Hader [10] presented data of times to failure for ARC-1 VHF communication transmitter-receivers of a single commercial airline. The times to failure were actually times to removal of units that are assumed to be failed. After the removal, the units were sent to maintenance and it turned out that some of the units were not failed after all. Time to failure for the confirmed failures will be represented by X, while time to removal of the units with unconfirmed failures is represented by Z. In addition the sample was censored at time $\tau = 630$ hours because the airline removed every unit which had been operated for that long. (This is a type I-censoring). There are $m = 218$ observations of X, $n = 107$ observations of Z and $r = 44$ censored observations. In the following these data will be referred to as the VHF-data.

Nonparametric estimates of $\tilde{S}_X(t) = 1 - \tilde{F}_X(t)$ and $\tilde{S}_X(t) = 1 - \tilde{F}_X(t)$ are given in Figure 3, computed by means of formulas in Lawless [7, Ch. 9.2]. These indicate that the condition for random signs censoring, $\tilde{F}_X(t) < \tilde{F}_Z(t)$, holds for these data at least for $t > 100$, while the situation is not that clear for $t < 100$. Still we assume that a random signs censoring model can be applied to the data.

Maximum likelihood estimates of the parameters c, s, ν and q are displayed in Table I, which also gives 95% a confidence interval for each parameter.

TABLE I

TABLE OF MAXIMUM LIKELIHOOD ESTIMATES OF THE PARAMETERS ν, c, s AND q FOR THE VHF-DATA IN THE WIENER PROCESS MODEL WITH CENSORING. IN ADDITION THE STANDARD DEVIATION FROM THE HESSIAN MATRIX AND 95% STANDARD POSITIVE INTERVALS ARE INCLUDED.

Param	Estimate	St deviation	Lower bound	Upper bound
ν	0.03412	0.003838	0.02737	0.04254
c	12.64	0.5780	11.56	13.83
s	10.64	0.6762	9.392	12.05
q	0.3230	0.02592	0.2760	0.3780

The confidence intervals for the levels c and s are slightly interfering, indicating that these levels could lie close to each other. This observation is in agreement with Figure 3 where the curves are close.

The parametric estimates of $\tilde{S}_X(t)$ and $\tilde{S}_Z(t)$ are also plotted in Figure 3. The fit appears to be rather bad, however, except for t large ($t > 400$). The reason is presumably that the inverse Gaussian distribution is not a good model for these data.

For comparison, we plotted also (Figure 4) the corresponding parametrically estimated curves for the repair alert model. Analysis of the VHF-data by means of a parametric repair alert is performed in [9] with X being exponentially distributed with parameter λ and the cumulative repair alert function given by $G(t) = t^{\beta}$. Maximum likelihood estimates of λ and β were calculated to be $\hat{\lambda} = 3.10 \cdot 10^{-3}$ and $\hat{\beta} = 4.44$, while q was estimated as 0.318. The fit to the data seems to be much better for this model as seen from Figure 4.

VII. CONCLUDING REMARKS

The non-identifiability of marginal distributions is well known in competing risks situations and the issue is well described, for example, in the book by Crowder [5]. The problem is apparent in reliability analyses, since the estimation of marginal failure distributions is of primary importance there. Random signs censoring is an interesting option in such studies, but like all other approaches this is again based on non-testable assumptions.

Instead of the Wiener process model studied in the present paper, we may use other types of processes, for example

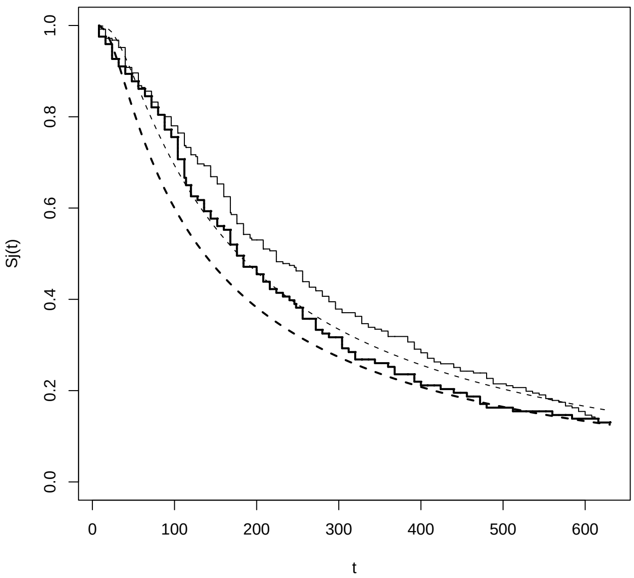

Fig. 3. Parametric estimated conditional survival functions $\hat{\tilde{S}}_X(t)$ (thin dashed line) and $\hat{\tilde{S}}_Z(t)$ (thick dashed line) for the basic models in the VHF-data. Plotted with the non-parametric estimates of $\tilde{S}_X(t)$ (thin line) and $\tilde{S}_Z(t)$ (thick line).

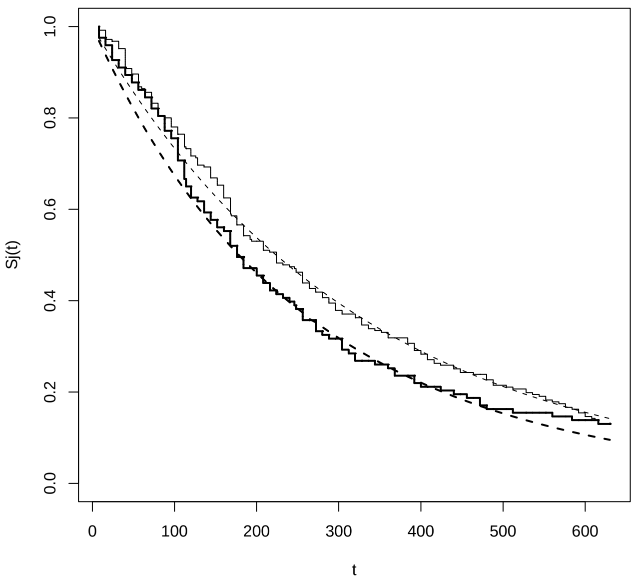

Fig. 4. Parametric estimated conditional survival functions $\hat{\tilde{S}}_X(t)$ (thin dashed line) and $\hat{\tilde{S}}_X(t)$ (thick dashed line) for the repair alert model with $f_X(x) = \lambda e^{-\lambda x}$ and $G(t) = t^{\beta}$ applied on the VHF-data [8]. Plotted with the non-parametric estimates of $\tilde{S}_X(t)$ (thin line) and $\tilde{S}_Z(t)$ (thick line)

gamma processes. An intuitive advantage of the latter processes is that they are strictly increasing, which seems more reasonable for a degradation process.

Skogsrud [11] considered several extensions of the Wiener process model presented here, obtained for example by letting the level s of PM be a random variable.

The model considered here can be extended to include the possibility of covariates. This can be done in a way similar to the one described by Aalen and Gjessing [1]. For example, we could let the level c of failure depend on the covariates. Also, we could let covariates influence the drift parameter ν.

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