The Trend-Renewal Process for Statistical Analysis of Repairable Systems

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The most commonly used models for the failure process of a repairable system are nonhomogeneous Poisson processes, corresponding to minimal repairs, and renewal processes, corresponding to perfect repairs. This paper introduces and studies a more general model for recurrent events, called the Trend-Renewal Process (TRP). The TRP is a time transformed renewal process, having both the ordinary renewal process and the nonhomogeneous Poisson process as special cases. Parametric inference in the TRP model is studied, with emphasis on the case where several systems are observed in the presence of a possible unobserved heterogeneity between systems.

KEY WORDS: Recurrent events; Poisson processes; Renewal processes; Heterogeneity;. Likelihood inference; Residual processes

1. INTRODUCTION

Statistical models for recurrent events which incorporate both time trends and renewal type behavior have recently been investigated in several papers. Such models are of interest for example in repairable systems reliability as a supplement to the widely used nonhomogeneous Poisson process (NHPP) and renewal process (RP) models, which strictly are applicable only under minimal repair and perfect repair conditions, respectively.

Berman and Turner (1992) and Lawless and Thiagarajah (1996) studied a class of models which can be viewed as extensions of the modulated renewal process of Cox (1972). A slightly different class, called the inhomogeneous gamma process, was suggested by Berman (1981), while a special case called the modulated power law process was introduced by Lakey and Rigdon (1992). The latter process has been studied further in Black and Rigdon (1996) and several other papers. A recent reference is Calabria and Pulcini (1999). The attractive feature of the inhomogeneous gamma processes and modulated power law processes is that they can be viewed as compromises between gamma renewal processes and (power law) nonhomogeneous Poisson processes.

In this paper we demonstrate how the basic idea behind Berman's inhomogeneous gamma process can be utilized in defining a more general class of models, called the trend-renewal process (TRP). Recall that Berman (1981) motivated

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the inhomogeneous gamma process as being the process $T_1, T_2, \ldots$ of observing only every κth event of a nonhomogeneous Poisson process with intensity function $\lambda(t)$. Defining $\Lambda(t) = \int_0^t \lambda(u)du$, Berman noted that this is equivalent to having $\Lambda(T_i) - \Lambda(T_{i-1})$ independent and identically distributed according to a gamma distribution with unit scale parameter and shape parameter κ. Using this as the definition of the inhomogeneous gamma process, it follows that κ can be any positive number.

The TRP as studied in this paper is simply a process $T_1, T_2, \ldots$ for which the $\Lambda(T_i) - \Lambda(T_{i-1})$, with $\Lambda(\cdot)$ defined as above, are i.i.d. with any positive distribution F (which later for convenience will be required to have expected value equal to 1.) The TRP class thus includes both the general NHPP, obtained if F is the unit exponential distribution, and the general renewal process, obtained if $\lambda(t)$ is a constant. Of course the TRP includes Berman's process by letting F be a gamma distribution. However, the TRP class is not generally included in, nor does it include, the modulated renewal process of Cox (1972) and the cited general models of Berman and Turner (1992) and Lawless and Thiagarajah (1996).

The main emphasis of the present paper is on statistical inference in parametric TRP models. We shall particularly be concerned with cases where several independent processes are observed simultaneously. In applications this means that sequences of recurrent events are observed for several individuals, for example failures of machines of the same type. Models for such situations may need to include individual effects, either observed or non-observed ones, for example caused by the installation of machines in different environments. Engelhardt and Bain (1987), Lawless (1987) and recently Ng and Cook (1999) consider unobserved heterogeneity in NHPP models, while Follmann and Goldberg (1988) and Aalen and Husebye (1991) treat heterogeneity in the RP case.

In the present paper we suggest corresponding models, called HTRP, which extend the TRP to cases involving non-observed heterogeneity. An interesting feature of the HTRP model is that, despite the rather simple structure, three important aspects of the possible behavior of a set of recurrent processes are taken care of: Poisson versus renewal behavior of inter-arrival times; stationarity versus nonstationarity for individual processes, and finally homogeneity versus heterogeneity between processes. These aspects constitute the three dimensions in a graphical description of the HTRP and its submodels as a cube, which we find useful for description in practical applications (see Figure 2). We also briefly indicate how observed covariates can be taken into account in a straightforward manner within the HTRP setting.

Other approaches to the analysis of recurrent events observed for several individuals are reviewed by Lawless (1995). In particular, Lawless and Nadeau (1995) study the problem of estimating mean functions for such cases.

The motivation for the present study has been modeling and analysis of failure data for repairable systems. Thus, although most of the models and methods studied will have wider applicability to phenomena involving recurrent events, we shall find it convenient to use the reliability terminology in the following. For a general treatment of repairable systems reliability we refer to the book by Ascher and Feingold (1984).

The literature on reliability contains a class of models called "imperfect repair" models, with the aim of filling the gap between the renewal process (perfect repair) and the NHPP (minimal repair). The classical model here is the one of Brown and Proschan (1983), which assumes that at the time of each failure a perfect repair occurs with probability p and a minimal repair occurs with probability $1-p$. In a recent paper, Dorado et al. (1997) suggest a general repair model which contains several known models of this kind as special cases. The statistical analyses of these models, however, often assume that extra information on the repair action is available at each failure in addition to the failure times, while the analyses presented in this paper use information only on the observed failure times.

The plan of this paper is as follows. Section 2 provides some basic notation and definitions. Then in Section 3 we proceed by defining the TRP and stating some main properties. The likelihood functions of the HTRP and its submodels are derived in Section 4. In particular it is shown how the submodels of the HTRP can be represented in a cube. Parametric modeling and inference in the HTRP is discussed in Section 5, while Section 6 is devoted to model checking based on residual processes which are ideally homogeneous Poisson processes. Section 7 contains three applications to data from repairable systems. Some final comments are given in Section 8.

2. REPAIRABLE SYSTEMS: NOTATION AND DEFINITIONS

Following Ascher and Feingold (1984), a repairable system is defined to be a system which, after failing to perform one or more of its functions satisfactorily, can be restored to fully satisfactory performance by any method, other than replacement of the entire system.

We consider repairable systems which are observed from time $t = 0$, with successive failure times denoted $T_1, T_2, \ldots$. An equivalent representation of the failure process is in terms of the counting process $\{N(t); t \geq 0\}$, where $N(t) =$ number of failures in $(0, t]$. It is assumed that simultaneous failures are not possible. The jumps of the process $\{N(t)\}$ are thus all of size 1.

An implicit assumption in the paper is that repair times equal 0. This is a reasonable assumption if repair times are negligible compared to times between failures, or if the time scale is taken to be the operation time of the system. Note that in the latter case, a marked point process may keep track of the repair times, and possibly additional information such as production conditions, maintenance and external conditions. However, marked processes will not be considered in this paper.

In order to mathematically describe models for repairable systems, we introduce some more notation from the theory of point processes. Let $\mathcal{F}_{t-}$ denote the history of the failure process up to, but not including, t. In this paper we shall always let $\mathcal{F}_{t-}$ be generated by $\{N(s) : 0 \leq s < t\}$ thus keeping record of all the failures strictly before time t. The $\mathcal{F}_{t-}$ could, however, contain more information, for example in a marked point process approach. Following Andersen et al. (1993, Section II.4) we define the conditional intensity $\phi(t|\mathcal{F}_{t-})$ by

$$\phi(t|\mathcal{F}_{t-}) = \lim_{\Delta t \downarrow 0} \frac{Pr(\text{failure in } [t, t+\Delta t)|\mathcal{F}_{t-})}{\Delta t}, \quad t > 0 \tag{1}$$

Thus, $\phi(t|\mathcal{F}_{t-})\Delta t$ is approximately the probability of failure in the time interval $[t, t+\Delta t)$, conditional on the experienced failure history before time t. Note that the so called ROCOF (rate of occurrence of failures, Ascher and Feingold 1984) is the corresponding unconditional intensity.

For a renewal process with interarrival distribution F (for short, RP(F)) it is well known that $\phi(t|\mathcal{F}_{t-}) = z(t - T_{N(t-)})$ where z is the hazard rate corresponding to the distribution F, and $t - T_{N(t-)}$ is the time since the last failure strictly before time t. Thus at each failure, the conditional intensity is returned to what it was at time 0, which means that the system is as good as new immediately after each failure (perfect repair).

On the other hand, an NHPP with intensity function $\lambda(t)$ (for short, NHPP($\lambda(\cdot)$)) has the conditional intensity $\phi(t|\mathcal{F}_{t-}) = \lambda(t)$. This means that the conditional intensity after a failure is exactly as if no failure had ever occurred. Thus at failures, the system is only restored to a condition where it is exactly as good (or bad) as it was immediately before the failure (minimal repair).

In this paper we study processes, either single or several independent similar processes, observed on (possibly different) time intervals of the form $[0, \tau]$. The observed failure times for a specific system are denoted $T_1, T_2, \ldots, T_{N(\tau)}$. The censoring time τ can in general be any stopping time with respect to the failure history. For a precise definition of stopping time we refer to Andersen et al. (1993, Section II.2). A nice informal discussion of censoring and stopping times is given by Aalen and Husebye (1991). Intuitively, a censoring time is a stopping time when the "decision" to stop at a time t depends only on the failures observed until time t, and not on those occurring later. Thus, stopping after a fixed number of failures (failure censoring) and stopping after a fixed time (time censoring) are examples of stopping times. An important consequence of censoring at stopping times is that the likelihood becomes the same as if censoring had occurred at some fixed time. Thus it follows from Andersen et al. (1993, Section II.7) that under our stated conditions the likelihood function for the observations from a single system is given by

$$L = \{\prod_{i=1}^{N(\tau)} \phi(T_i|\mathcal{F}_{T_i-})\} \exp(-\int_0^\tau \phi(u|\mathcal{F}_{u-})du) \tag{2}$$

3. THE TREND-RENEWAL PROCESS

The trend-renewal process generalizes the following well known property of the NHPP: If $T_1, T_2, \ldots$ is an NHPP($\lambda(\cdot)$), then the time-transformed stochastic process $\Lambda(T_1), \Lambda(T_2), \ldots$ is HPP(1). Here $\Lambda(t) = \int_0^t \lambda(u)du$ is the cumulative intensity, and HPP(ν) denotes a homogeneous Poisson process with intensity ν.

We define the trend-renewal process (TRP) simply by allowing the above HPP(1) to be any renewal process RP(F) where the distribution F has expected value equal to 1. Thus, in addition to the intensity function $\lambda(t)$ we need to specify a distribution function F. Formally we can define the process TRP($F, \lambda(\cdot)$) as follows.

Definition.

Let $\lambda(t)$ be a nonnegative function defined for $t \geq 0$, satisfying $\Lambda(t) \equiv \int_0^t \lambda(u)du < \infty$ for each $t \geq 0$ and $\Lambda(\infty) = \infty$. Let further F be a positive distribution (i.e. $F(0) = 0$) with expected value 1. The process $T_1, T_2, \ldots$ is called $\mathrm{TRP}(F, \lambda(\cdot))$ if the time-transformed process $\Lambda(T_1), \Lambda(T_2), \ldots$ is $\mathrm{RP}(F)$, i.e. if the $\Lambda(T_i) - \Lambda(T_{i-1}),\ i = 1, 2, \ldots$, are i.i.d. with distribution function F.

We shall call F the renewal distribution and $\lambda(\cdot)$ the trend function of the TRP. Figure 1 illustrates the definition. Note that for an $\mathrm{NHPP}(\lambda(\cdot))$, the $\mathrm{RP}(F)$ on the lower time axis is replaced by $\mathrm{HPP}(1)$. Thus the TRP generalizes the NHPP.

For consistency of notation, we shall find it convenient in the sequel to represent renewal processes by the notation $\mathrm{RP}(F, \nu)$, where F is a distribution with expected value 1 and the constant ν equals the inverse expected value of the interarrival distribution. This means that a renewal process with positive interarrival distribution G having finite expectation is represented as $\mathrm{RP}(F, \nu)$, where ν is the inverse expectation of G and $F(\cdot) = G(\cdot/\nu)$. With this convention we have $\mathrm{TRP}(F, \nu) = \mathrm{RP}(F, \nu)$ when ν is a constant. In particular this shows that the TRP includes all RP with positive interarrival distribution and finite expectation.

Note that the requirement that F has expected value 1 in the definition of $\mathrm{TRP}(F, \lambda(\cdot))$ is made for convenience. The scale is then absorbed by the trend function $\lambda(t)$. In the following it will be tacitly assumed when needed that the hazard function $z(t)$ and density function $f(t)$ corresponding to F exist.

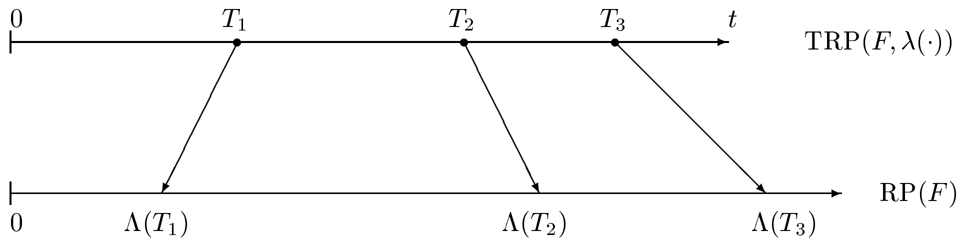

Figure 1. The defining property of the trend-renewal process.

The inhomogeneous gamma process (Berman 1981) corresponds to F being a gamma-distribution in the definition of the TRP. The modulated power law process (Lakey and Rigdon 1992) is the further specialization where the trend function is given by

$$\lambda_{pl}(t) = abt^{b-1} \tag{3}$$

for parameters $a, b > 0$, the so called power law function. Note that a gamma-distribution with expected value 1 can be described by a single parameter (12). Thus the modulated power law process can be given in terms of three parameters. Berman (1981) and Lakey and Rigdon (1992) use an equivalent parameterization where the renewal distribution is a gamma-distribution with scale parameter equal to 1.

The conditional intensity function (1) for the TRP$(F, \lambda(\cdot))$ is obtained from the following derivation.

$$\begin{aligned} \phi(t|\mathcal{F}_{t-})\Delta t &\doteq P(\text{failure in } [t, t+\Delta t)|\mathcal{F}_{t-}) \\ &= P(\text{failure in } RP(F) \text{ in } [\Lambda(t), \Lambda(t+\Delta t))|\mathcal{F}_{t-}) \\ &\doteq P(\text{failure in } RP(F) \text{ in } [\Lambda(t), \Lambda(t)+\lambda(t)\Delta t)|\mathcal{F}_{t-}) \\ &= z(\Lambda(t) - \Lambda(T_{N(t-)}))\lambda(t)\Delta(t) \end{aligned}$$

so that

$$\phi(t|\mathcal{F}_{t-}) = z(\Lambda(t) - \Lambda(T_{N(t-)}))\lambda(t) \tag{4}$$

Thus the conditional intensity function of the TRP is a product of one factor, $\lambda(t)$, which depends on the age t of the system and one factor which depends on the (transformed) time from the last previous failure. In fact time since last failure is measured on the scale $\Lambda(t)$ which depends on the cumulative aging effect. This is in contrast to Cox's modulated renewal process (Cox 1972) for which the conditional intensity is of the form $z(t - T_{N(t-)})\lambda(t)$.

As a motivation for the TRP model, suppose that failures of a particular system correspond to replacement of a major part, for example the engine of a tractor (see Example 7.2), while the rest of the system is not maintained. Then if the rest of the system is not subjected to wear, a renewal process would be a plausible model for the observed failure process. In the presence of wear, however, an increased replacement frequency is to be expected. This is achieved in a TRP model by accelerating the internal time of the renewal process according to a time transformation $\Lambda(t) = \int_0^t \lambda(u)du$ which represents the cumulative wear. Thus in this case it would be natural to call $\lambda(\cdot)$ the wear function rather than the trend function.

As indicated by the above, the TRP model has some similarities to accelerated failure time models (see e.g. Crowder et al. 1991, Section 4.2). Recall that an accelerated life time model is given in terms of a baseline lifetime distribution function $F(\cdot)$ such that the lifetime of a particular individual has distribution function $F(\nu t)$ for some value of $\nu > 0$. (Usually, ν is a function of a covariate vector.) The intuitive idea is that time is accelerated by a factor ν, which may vary from individual to individual. More generally, in a so called transformation model (Andersen et al. 1993, Section VII.5) the time acceleration is defined through an increasing function $\Lambda(t)$, leading to the life time distribution function $F(\Lambda(t))$. Now for the TRP, the distribution function of T_i, the time of the ith failure, is $F^{(i)}(\Lambda(t))$, where $F^{(i)}$ is the $i$fold convolution of F by itself. Thus the TRP model implies a transformation model for each specific failure time.

We close this section by presenting two limit results for the TRP, which follow from the general theory of renewal processes. Let as before $N(t)$ denote the number of failures in $(0, \tau]$ and define $V(t) = E[N(t)]$. Assume furthermore that $\Lambda(t)$ is strictly increasing.

The definition of TRP$(F, \lambda(\cdot))$ states that $N^*(u) \equiv N(\Lambda^{-1}(u))$, $u > 0$, is a renewal process, RP$(F, 1)$. Moreover, if we define $V^*(u) = E[N^*(u)]$ then $V^*(u) = V(\Lambda^{-1}(u))$. Standard asymptotic results for renewal processes (Ross

1983, Section 3.3) imply that $N^*(u)/u$ tends to 1 almost surely, and that $V^*(u)/u \to 1$ as $u \to \infty$. Now substituting $t = \Lambda^{-1}(u)$ gives the following limit results for the TRP,

$$\frac{N(t)}{\Lambda(t)} \to 1 \text{ a.s. }, \frac{V(t)}{\Lambda(t)} \to 1 \text{ as } t \to \infty$$

Thus we may, at least asymptotically, think of $\Lambda(t)$ as the expected number of failures until time t.

4. STATISTICAL INFERENCE IN THE TRP MODEL: LIKELIHOOD FUNCTION

4.1 Likelihood for a Single TRP

Suppose first that a single system has been observed in $[0, \tau]$, for a stopping time τ, with failures at $T_1, T_2, \ldots, T_{N(\tau)}$. If a TRP$(F, \lambda(\cdot))$ is used as a model, then substitution of (4) into (2) gives the likelihood

$$L = \{\prod_{i=1}^{N(\tau)} z(\Lambda(T_i) - \Lambda(T_{i-1}))\lambda(T_i)\} \exp\{-\int_0^\tau z(\Lambda(u) - \Lambda(T_{N(u-)}))\lambda(u)du\} \tag{5}$$

where $z(\cdot)$ is the hazard rate corresponding to F. Equivalently, if f is the density function corresponding to F, we can write the likelihood as

$$L = \{\prod_{i=1}^{N(\tau)} f(\Lambda(T_i) - \Lambda(T_{i-1}))\lambda(T_i)\}\{1 - F(\Lambda(\tau) - \Lambda(T_{N(\tau)}))\} \tag{6}$$

The latter form of the likelihood follows since the conditional density of T_i given $T_1 = t_1, \ldots, T_{i-1} = t_{i-1}$ is $f(\Lambda(t_i) - \Lambda(t_{i-1}))\lambda(t_i)$, and the probability of no failures in the time interval $(T_{N(\tau)}, \tau]$, given $T_1, \ldots, T_{N(\tau)}$, is $1 - F(\Lambda(\tau) - \Lambda(T_{N(\tau)}))$.

4.2 Several Systems, the HTRP Model

Suppose that m systems are observed, where the jth system $(j = 1, 2, \ldots, m)$ is observed in the time interval $[0, \tau_j]$. For the jth system, let N_j denote the number of failures that occur during the observation period, and let the specific failure times be denoted $T_{1j} < T_{2j} < \cdots < T_{N_j j}$.

Even if the systems are considered to be of the same type, they may well exhibit different probability mechanisms. For example, systems may be used under varying environmental or operational conditions. To cover such cases we shall assume that failures of the jth system follow the process TRP$(F, \lambda_j(\cdot))$, $j = 1, \ldots, m$, where the renewal function F is fixed and differences between systems are modeled by letting the trend functions $\lambda_j(t)$ vary from system to system. The assumption of a fixed F corresponds to what is done in the NHPP case, where F is the unit exponential distribution.

Assuming that systems work independently of each other, we obtain from (6) the full likelihood $L \equiv \prod_{j=1}^{m} L_j$ where

$$L_j = \{\prod_{i=1}^{N_j} f(\Lambda_j(T_{ij}) - \Lambda_j(T_{i-1,j}))\lambda_j(T_{ij})\}\{1 - F(\Lambda_j(\tau_j) - \Lambda_j(T_{N_j j}))\} \tag{7}$$

As an example of the use of (7), assume that differences between system performances can be attributed to an observable covariate vector $\mathbf{x}$, and that the trend $\lambda_j(t)$ for system j is represented by a proportional hazards model with

$$\lambda_j(t) = g(\mathbf{x}_j)\lambda(t), \quad j = 1, \ldots, m \tag{8}$$

Here $\lambda(\cdot)$ is a basic trend function common to all systems, while g is a function of the covariate vector $\mathbf{x}_j$ of system j. The special cases of this model corresponding to NHPP and RP are studied, respectively, by Lawless (1987), and Follmann and Goldberg (1988).

In addition to observable covariates there may in practice be an unobserved heterogeneity between the systems (in the literature also called "individual effects" or "frailty"). A common way of incorporating such heterogeneity is to modify (8) to $\lambda_j(t) = a_j g(\mathbf{x}_j)\lambda(t)$ where the a_j are unobservable (positive) random variables taking values independently across systems (Andersen et al. 1993, Chapter IX). Thus, modeling and analysis of heterogeneity can be viewed as an example of the Empirical Bayes technique (see e.g. Maritz and Lwin 1989).

For simplicity we shall in this paper restrict attention to the case with no observed covariates, and instead concentrate on unobserved heterogeneity. Note that an extension to include the covariate case, using for example a Cox-type model $g(\mathbf{x}) = \exp\{\boldsymbol{\beta}'\mathbf{x}\}$ where $\boldsymbol{\beta}$ is a vector of unknown parameters, would be rather straightforward in the parametric setting of this paper.

In the following we assume the model

$$\lambda_j(t) = a_j\lambda(t) \tag{9}$$

where the a_j are independently distributed according to a common probability distribution H, say, and where for convenience we assume that the expected value of a_j equals 1. Thus in (9), $\lambda(\cdot)$ is regarded as a basic trend function, while the a_j represent a possibly different failure intensity "level" for each system, averaging to 1. The special case when $a_j = 1$ with probability 1 will be referred to as the "no heterogeneity" case.

For given values of the a_j the likelihood for the jth system is, by (7),

$$L_j(a_j) = \{\prod_{i=1}^{N_j} f(a_j(\Lambda(T_{ij}) - \Lambda(T_{i-1,j})))a_j\lambda(T_{ij})\}\{1 - F(a_j(\Lambda(\tau_j) - \Lambda(T_{N_j j})))\} \tag{10}$$

Table 1. *The seven submodels of $HTRP(F, \lambda(\cdot), H)$. 'exp' means the unit exponential distribution, '1' means the distribution degenerate at 1. The third column contains references to work on the corresponding models or special cases of them.*

Submodel	*HTRP-formulation*	*Some relevant references*
$HPP(\nu)$	$HTRP(\exp, \nu, 1)$	*Standard*
$RP(F, \nu)$	$HTRP(F, \nu, 1)$	*Standard*
$NHPP(\lambda(\cdot))$	$HTRP(\exp, \lambda(\cdot), 1)$	*Ascher and Feingold (1984)*
		Lawless (1987)
$TRP(F, \lambda(\cdot))$	$HTRP(F, \lambda(\cdot), 1)$	*Berman (1981)*
		Black and Rigdon (1996)
$HHPP(\nu, H)$	$HTRP(\exp, \nu, H)$	*Proschan (1963)*
		Bain and Wright (1982)
$HRP(F, \nu, H)$	$HTRP(F, \nu, H)$	*Follmann and Goldberg (1988),*
		Aalen and Husebye (1991)
$HNHPP(\lambda(\cdot), H)$	$HTRP(\exp, \lambda(\cdot), H)$	*Engelhardt and Bain (1987)*
		Lawless (1987),
		Ng and Cook (1999).

and the full likelihood is, since the a_j are unobservable,

$$L = \prod_{j=1}^{m} \int L_j(a_j) dH(a_j) \tag{11}$$

We shall use the notation HTRP$(F, \lambda(\cdot), H)$ for the model with likelihood (11). Here the renewal distribution F and the heterogeneity distribution H are distributions corresponding to positive random variables with expected value 1, while the basic trend function $\lambda(t)$ is a positive function defined for $t \geq 0$.

A useful feature of the HTRP model is that several important models for repairable systems are easily represented as submodels. With the notation HPP, NHPP, RP and TRP used as before, we define corresponding models with heterogeneity (9) by putting an H in front of the abbreviations. Specifically, from a full model, HTRP$(F, \lambda(\cdot), H)$, we can identify the seven submodels described in Table 1. The table also contains relevant references to the literature.

The HTRP and the seven submodels can be represented in a cube, as illustrated in Figure 2. Each vertex of the cube represents a model, and the lines connecting them correspond to changing one of the three "coordinates" in the HTRP-notation. It is seen that going to the right corresponds to introducing a time trend, going upwards corresponds to entering a non-Poisson case and finally that going backwards (inwards) corresponds to introducing heterogeneity. In analysis of data by parametric HTRP models we use the cube to give an easy presentation of maximum log-likelihood values for the different models (Figure 2). In the following we shall therefore use the notion of log-likelihood cube.

5. PARAMETRIC INFERENCE IN THE HTRP MODEL

A general parametric HTRP model is obtained through defining parametric functions $F(y;\theta), \lambda(t;\theta)$, $H(a;\theta)$ in the likelihood (10-11). Here $\theta \in \Theta$ is usually a vector which can be decomposed as $\theta = (\theta_F, \theta_\lambda, \theta_H)$ where F depends on θ only through θ_F etc. In order to uniquely identify the seven submodels of Table 1, the parameterization should be such that there is a unique value of θ_F corresponding to F being the unit exponential distribution, a unique value of θ_H corresponding to H being degenerate at 1, and finally that to each constant $c > 0$ there should correspond a unique value of the parameter θ_λ representing the constant function c.

In Example 5.1 and in the data examples in Section 7 the distributions F and H are gamma or Weibull distributions with expected value 1. We distinguish these distributions by using subscripts g and w, respectively, and note that the densities are given by

$$f_g(x|\gamma) = (\Gamma(1/\gamma))^{-1}\gamma^{-1/\gamma}x^{1/\gamma-1}e^{-x/\gamma}, \quad x > 0 \tag{12}$$

$$f_w(x|\beta) = \beta^{-1}(\Gamma(\beta+1))^{1/\beta}x^{1/\beta-1}\exp(-(\Gamma(\beta+1)x)^{1/\beta}), \quad x > 0 \tag{13}$$

where $\gamma > 0,\ \beta > 0$. Putting $\gamma = 1$ and $\beta = 1$, respectively, give unit exponential distributions. Moreover, the limiting cases $\gamma = 0$ and $\beta = 0$ correspond to degenerate distributions at the value 1. Thus the given requirements for HTRP parameterization are satisfied by extending the parameter spaces to $\gamma \geq 0$ and $\beta \geq 0$.

A commonly used parameterization for the trend function $\lambda(\cdot)$ is the power law function (3). Here the values $b = 1$, $a = c$ give the required function constant at c.

Given a model HTRP$(F, \lambda(\cdot), H)$ and given a set of data $\{T_{ij}\}$ we may now fit the full model and the submodels by the maximum likelihood method. The log-likelihood value for each model can then be recorded in the log-likelihood cube (e.g. Figure 2). The cube suggests three main types of statistical tests to be performed, corresponding to each of the three dimensions. The first are tests for the null hypothesis that F is the unit exponential distribution. These are tests for checking the Poisson-property. Next we may test the null hypothesis that the trend function $\lambda(\cdot)$ is constant in time. This is known as trend testing. Finally, we may test the null hypothesis that the heterogeneity distribution H is degenerate at 1, which leads to tests for homogeneity across systems.

It follows that the selection of a model in the log-likelihood cube is typically the result of multiple tests. Thus one should be careful in the interpretation of P-values of single tests. We will not, however, pursue this issue in the present paper.

Comparison of nested submodels (i.e. models for which there is a directed path in the log-likelihood cube from the one to the other) can at least in principle be performed by likelihood ratio testing. The common way is to compute the likelihood ratio statistic given by twice the difference in log-likelihood and comparing to the chi-square distribution with degrees of freedom corresponding to the difference in number of free parameters. This approach relies on asymptotic theory, and is valid under specific conditions which need to be examined. General asymptotic results for parametric inference in counting process models are derived in Andersen et al. (1993, Chapter VI). Typically, these results are relevant for parametric TRP models without heterogeneity in the cases where the number m of processes tends to infinity, while the number of failures per system remains small or moderate. Moreover, the results may be applied if the number of systems is small or moderate while the intensity process is increasing to infinity. Asymptotic results for a single or small number of systems, when the censoring time tends to infinity, can also be derived using the approach of Svensson (1990).

For cases with heterogeneity between systems, asymptotics is valid only if the number m of systems tends to infinity. A general concern in analysis of heterogeneity models is, however, that the special case of "no heterogeneity" corresponds to some parameter falling on the boundary of the parameter space, making standard asymptotic results for regular models no longer valid. In the setting of the present section, this happens if the value of θ_H corresponding to a distribution degenerate at 1 is a boundary point. As we have seen, this is the case when H has the gamma or Weibull distributions of (12-13). A classical result by Chernoff (1954) states that in such cases the likelihood ratio statistic has an asymptotic distribution which is 0 with probability 1/2 and is chi-squared with probability 1/2. On the other hand, Aalen and Husebye (1991) and Andersen et al. (1993, Chapter IX) demonstrate how it may be possible to modify the parameter space to include boundary points as interior points in an extended parameter space. Although some of the original interpretation of the model may then be lost, this allows likelihood ratio testing to be done in the usual way. While such an approach is also possible for certain parametric HTRP models, it will not be studied in the present paper.

Regarding asymptotics, we note that a simulation study reported by Black and Rigdon (1996) indicated that the usual Gaussian approximations to standard deviations of parameter estimates may not be adequate for small and moderate samples for the modulated power law process. Thus alternative methods like bootstrapping (Efron and Tibshirani 1993) may be useful to improve the accuracy of inferences. In fact, the TRP is by its definition very well suited for bootstrap simulations. The obvious way of generating bootstrap samples from a $\text{TRP}(F, \lambda(\cdot))$ is to first simulate the renewal process $\text{RP}(\hat{F}, 1)$ and then transform arrival times by the inverse mapping of $\hat{\Lambda}(\cdot)$. Here the hats indicate functions estimated from the observed data. In Example 7.3 we have used bootstrapping to compute P-values of tests for model fit. A more thorough study of bootstrapping in TRP models is, however, left to a forthcoming paper.

An S-Plus library, called REIDAR (=REpaIrable systems Data Analysis Routines), containing routines for analysis of repairable systems, has been developed (Xxx 1999). In particular, the program contains routines for graphical analyses, parametric estimation in HTRP and the corresponding submodels, trend testing and model checking, for various parameterizations of F, H and $\lambda(\cdot)$. REIDAR computes parameter estimates in HTRP models by maximizing (11) using numerical integration and maximization. In most cases standard S-plus functions for integration (adaptive 15 point Gauss-Kronod quadrature) and minimization (quasi Newton) are used.

5.1 Example: The inhomogeneous gamma process with several systems and unobserved heterogeneity

In this example we study a generalization of the inhomogeneous gamma process of Berman (1981) to the case with several systems and a possible (gamma distributed) heterogeneity. With notation as introduced earlier, this is the model $\text{HTRP}(F_g, \lambda(\cdot), H_g)$, where $\lambda(\cdot)$ is some given parametric trend function.

Assume that m systems are observed under failure censoring, where the jth system is observed until a number n_j of failures have occurred ($j = 1, \ldots, m$). For simplicity of writing, put $K_{ij} = \Lambda(T_{ij}) - \Lambda(T_{i-1,j})$ where $T_{0j} = 0$. Then the likelihood for the jth system, given the value a_j, is obtained from (10) with f_g from (12) substituted for f,

$$L_j(a_j) = \{\prod_{i=1}^{n_j} K_{ij}^{1/\gamma-1}\lambda(T_{ij})\}\frac{a_j^{n_j/\gamma}\gamma^{-n_j/\gamma}}{(\Gamma(1/\gamma))^{n_j}}\exp\{-a_j\Lambda(T_{n_j j})/\gamma\} \tag{14}$$

The unconditional value L_j is found by taking the expected value of $L_j(a_j)$ when a_j has density given by (12) with parameter now denoted δ. This expectation can be computed analytically to be,

$$L_j = \{\prod_{i=1}^{n_j} K_{ij}^{1/\gamma-1}\lambda(T_{ij})\}\frac{\gamma^{-n_j/\gamma}}{(\Gamma(1/\gamma))^{n_j}}\frac{\delta^{-1/\delta}}{\Gamma(1/\delta)}\frac{\Gamma(n_j/\gamma+1/\delta)}{(1/\delta+(1/\gamma)\Lambda(T_{n_j j}))^{n_j/\gamma+1/\delta}} \tag{15}$$

The parameters of the model HTRP$(F_g, \lambda(\cdot), H_g)$ are estimated by maximizing $L = \prod L_j$ with respect to γ, δ and the parameters of $\lambda(\cdot)$. It may then happen that the maximum occurs at $\delta = 0$ on the boundary of the parameter space. This is the case in Example 7.2 where the present model and its submodels are fitted to real data. Note that the limit of (15) as $\delta \to 0$ is $L_j(1)$ in (14).

In order to check a conclusion that $\hat{\delta} = 0$ really is the MLE of δ we may compute the (right) derivative at $\delta = 0$ of $\log L = \sum \log L_j$ as a function of δ. Using asymptotic formulas for the digamma function $d[\log\Gamma(z)]/dz$ (Abramowitz and Stegun 1965) we arrive at

$$\frac{\partial \log L}{\partial \delta}|_{\delta=0} = (2\gamma^2)^{-1}[\sum_{j=1}^{m}(n_j - \Lambda(T_{n_j j}))^2 - n\gamma] \tag{16}$$

where $n = \sum_{j=1}^m n_j$. Now the (right) derivative of the profile likelihood of δ at $\delta = 0$ is given by (16) if we substitute maximum likelihood estimates $\hat{\gamma}_0$ for γ and $\hat{\Lambda}_0(\cdot)$ for $\Lambda(\cdot)$ under the model when $\delta = 0$. A negative value of (16) then implies a local maximum of the profile likelihood at $\delta = 0$, indicating that $\hat{\delta} = 0$. On the other hand, a positive value of (16) implies that $\hat{\delta}$ is strictly positive.

6. MODEL CHECKING

Suppose a parametric model HTRP$(F, \lambda(\cdot), H)$ has been fitted. We shall present a general model checking procedure based on so-called residual processes of the fitted model (Andersen et al. 1993, p. 499). Our approach is much similar to the use of Cox-Snell residuals in Cox regression (Cox and Snell 1968, Kay 1977).

Consider first a single process TRP$(F, \lambda(\cdot))$. It follows from general theory of point processes (Brémaud 1981, Section II.6) that the process of integrated conditional intensities, i.e. the process $\int_0^{T_1}\phi(t|\mathcal{F}_{t-})dt$, $\int_0^{T_2}\phi(t|\mathcal{F}_{t-})dt, \ldots$ is HPP(1). This is in fact seen directly in our case since by (4) the corresponding interarrival times are given by

$$\int_{T_{i-1}}^{T_i}\phi(t|\mathcal{F}_{t-})dt = \int_{T_{i-1}}^{T_i} z(\Lambda(t) - \Lambda(T_{i-1}))\lambda(t)dt = \int_0^{\Lambda(T_i)-\Lambda(T_{i-1})} z(y)dy \tag{17}$$

These random variables are clearly independent and unit exponentially distributed since the $\Lambda(T_i) - \Lambda(T_{i-1})$ by definition are independent with distribution F and $z(\cdot)$ is the hazard rate of F.

Consider next several systems modeled by HTRP$(F, \lambda(\cdot), H)$, and let the observations T_{ij} be as defined in the beginning of Section 4.2. Suppose, moreover, that the values a_j of (9) are given. Then for each system $j = 1, 2, \ldots, m$ we define

$$R_{ij} = \int_0^{a_j(\Lambda(T_{ij}) - \Lambda(T_{i-1,j}))} z(y)dy, \quad i = 1, 2, \ldots, N_j, \tag{18}$$

and for the cases where $T_{N_j j} < \tau_j$, we also define $R_{N_j+1,j} = \int_0^{a_j(\Lambda(\tau_j) - \Lambda(T_{N_j,j}))} z(y)dy$

As for (17) it follows that the R_{ij} for $i \le N_j$ constitute a set of independent unit exponential variables. Thus for each system $j = 1, \ldots, m$ we can construct an HPP(1) with $R_{1j}, R_{2j}, \ldots, R_{N_j j}$ defining successive interevent times, and when $T_{N_j j} < \tau_j$ with a censored interevent time $R_{N_j+1,j}$ added at the end.

The idea is now to define observable residuals $\{\hat{R}_{ij}\}$ by substituting estimates of unknown parameters and reconstructed values for the a_j in the above defined R_{ij}. The $\{\hat{R}_{ij}\}$ are then taken as interevent times of residual processes defined for each system j in the same manner as described for the R_{ij} in the previous paragraph. If the model is correct, the observed residual processes are expected to behave like independent HPP(1) processes.

The use of residuals and residual processes in model checking is demonstrated for three different data sets in Section 7. Typically, one might check (i) the distribution of the residuals $\hat{R}_{ij}$ with respect to departures from the unit exponential distribution, (ii) the possible presence of time trend in residuals within the individual systems j, e.g. whether the residuals $\hat{R}_{ij}$ tend to increase or decrease with failure number i, and (iii) autocorrelation of the residuals $\hat{R}_{ij}$ within systems.

For graphical check of (i) and (ii) we have used Total Time on Test plots (TTT-plots). These were introduced by Barlow and Campo (1975) and Barlow and Davis (1977) for independent failure data and recurrent events data, respectively. A unified treatment is given by Andersen et al. (1993, Section VI.3.2). Briefly, suppose that a possibly varying number of independent HPP processes are observed from time $t = 0$, where $Y(t)$ is the number of observed processes at time t, $t \ge 0$. Let further $S_1 < S_2 < \ldots < S_K$ be the jump times of the superimposed process, i.e. the union of the individual processes. Define $R(t) = \int_0^t Y(s)ds$. Then it can be shown that the process $R(S_1), R(S_2), \ldots$ is HPP(1), and hence that the scaled TTT-statistic $(R(S_1)/R(S_K), R(S_2)/R(S_K), \ldots, R(S_{K-1})/R(S_K))$ is distributed as the ordering of $K-1$ i.i.d. uniform random variables on $[0, 1]$. The TTT-plot is a plot of the points $(k/K, R(S_k)/R(S_K))$, $k = 1, \ldots, K$. Under the given HPP assumptions we therefore expect the TTT-plot to fall near the diagonal of the unit square.

For a numerical verification of the nearness of the TTT-plot to the diagonal we suggest computing the Anderson-Darling statistic (in the following called the AD-statistic) for testing uniformity on $[0, 1]$ of the data $R(S_k)/R(S_K)$; $k =$

$1, \ldots, K-1$ (Anderson and Darling 1954). Under the above given assumptions this statistic will have the asymptotic distribution derived by Anderson and Darling (1954), which we shall call the AD-distribution. The choice of the Anderson-Darling statistic may seem arbitrary, but simulations by Kvaløy and Lindqvist (1998) have shown that this leads to a test with adequate power versus a variety of alternatives.

The TTT-statistic and -plot, as described in general above, are used as follows for checking the distribution of the residuals (point (i) above). For the plot we let $Y(t)$ for each $t > 0$ be the number of residuals $\hat{R}_{ij}$ which are less than t. This corresponds to defining for each pair (i, j) a process started at time 0 and stopped by an event at time $\hat{R}_{ij}$ if $i \le N_j$ and by a censoring (no event) if $i = N_j + 1$. The resulting plot is the TTT-plot of Barlow and Campo (1975). Although the observed residuals are not exactly exponentially distributed, the plot is expected to fall near the diagonal when the model is correct.

Numerical verification of nearness of a TTT-plot to the diagonal can be performed as described in the general case above using the AD-statistic. Since we base ourselves on estimated parameters in the computation of the residual processes, P-values based on the AD-distribution should not be trusted, however. We suggest using a parametric bootstrapping to estimate the true distributions of the AD-statistics, and compute estimated P-values from these. An illustration is given in Example 7.3.

Checking for trend in the residual processes (point (ii) above) is done in a way similar to the one described above. Now we construct the TTT-plot on the basis of the m residual processes, one for each system. We then define $Y(t)$ to be the number of the m processes which are not yet censored at time t. This leads to the TTT-plot by Barlow and Davis (1977). As above we can use the AD-statistics to verify model fit.

In order to detect a possible autocorrelation in the residual processes (point (iii) above), we suggest plotting pairs of (log)residuals $((\log)\hat{R}_{ij}, (\log)\hat{R}_{i+k,j})$ from the same system at different lags k (see Crowder et al. 1991, p. 162). In two of the examples of Section 7 we have, moreover, computed estimates of the autocorrelation function by the formula

$$\hat{\rho}(k) = \frac{1}{m} \sum_{j=1}^{m} \left[\frac{N_j}{N_j - k} \frac{\sum_{i=1}^{N_j - k} (\hat{R}_{i+k,j} - \overline{R}_j)(\hat{R}_{ij} - \overline{R}_j)}{\sum_{i=1}^{N_j} (\hat{R}_{ij} - \overline{R}_j)^2} \right] \tag{19}$$

where the $\hat{R}_{ij}$, $i = 1, 2, \ldots$ are the uncensored residuals and $\overline{R}_j$ is the average of the uncensored residuals corresponding to system j $(j = 1, \ldots, m)$. The resulting values may, when appropriate, be compared to the asymptotic standard deviation under the assumption of independence, which is $m^{-1}\sqrt{\sum_{j=1}^{m} N_j^{-1}}$, assuming that all the N_j are large (Lehmann 1999, Example 5.2.6). For cases with small or moderate number of failures per system we suggest using bootstrapping for testing against autocorrelation. This is illustrated in Example 7.3.

It remains to describe how we find reconstructed values $\hat{a}_j$ of the a_j. In the case of no heterogeneity we put $a_j \equiv 1$. For the heterogeneous case where the a_j are unobserved, we use a Bayesian approach similar to the ones of Lawless (1987) and Follmann and Goldberg (1988). Thus for the jth system, modelled by $\mathrm{TRP}(F, a_j\lambda(\cdot))$, suppose a_j is the only unknown parameter and consider H as its prior distribution. The likelihood function is then $L_j(a_j)$ in (10), and the posterior distribution of a_j is

$$\Pi(a_j|T_{1j}, T_{2j}, \ldots, T_{N_j j}) \propto h(a_j)L_j(a_j) \tag{20}$$

The expectation of this posterior distribution, with estimated parameters used in F, $\lambda(\cdot)$ and H, is used as the reconstructed value $\hat{a}_j$ of the heterogeneity parameter a_j.

In practice it may be difficult to interpret residual plots as described in the present section. Thus one may wish instead to consider nonparametric Nelson-Aalen plots (Andersen et al. 1993) based on the observed times T_{ij} in order to judge a parametrically estimated $\Lambda(\cdot)$. In the same manner one may wish to plot empirical distribution functions based on the transformed interarrival times $\hat{\Lambda}(T_{i+1}) - \hat{\Lambda}(T_i)$ for comparison with a parametrically estimated F.

6.1 Example - continued from 5.1

In the present case it is straightforward to show that the posterior distribution of a_j given $T_{1j}, \ldots, T_{n_j j}$ is a gamma distribution with expected value

$$\frac{\gamma + n_j\delta}{\gamma + \delta\Lambda(T_{n_j j})} \tag{21}$$

This defines our estimate $\hat{a}_j$ of a_j when estimates are substituted for δ, γ and $\Lambda(\cdot)$. It is interesting to note that we can write (21) as a convex combination of 1, which corresponds to the value of a_j we would have in the case of no heterogeneity, and $n_j/\Lambda(T_{n_j j})$ which is the maximum likelihood estimator of a_j based on the likelihood (14) for the jth system.

7. EXAMPLES

7.1 Proschan's Data

In order to compare our methods to approaches by other authors we reconsider the well known airconditioner failure data of Proschan (1963). Proschan organized the data in $m = 17$ systems with a total of 213 failures, with the number of failures for each system varying from $n = 3$ to $n = 30$. The 1963 paper by Proschan is believed to be the first to treat heterogeneity in repairable systems data. His point of departure was a theoretical result that independent exponential variables with different parameters, when analysed as i.i.d. variables, reveal a decreasing hazard rate. In the data he found that the individual systems are each well described by HPP(ν) models, but with ν possibly varying from system to system. By considering the collection of all the interfailure times and using a test for constant hazard versus decreasing hazard, he concluded that the individual ν are different. Follmann and Goldberg (1988) confirmed

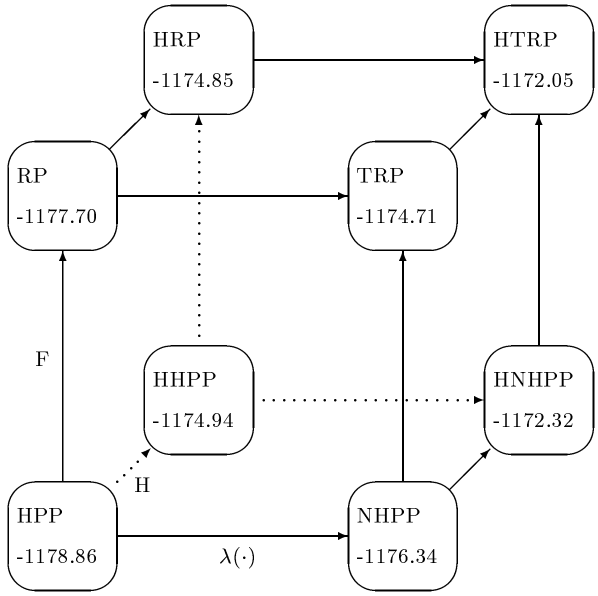

Figure 2. The log-likelihood cube of the HTRP($F_w, \lambda_{pl}(\cdot), H_g$) model for Proschan's data

Proschan's conclusion by fitting a model which is essentially our HRP(F_w, ν, H_g) and concluded that the submodel HHPP(ν, H_g) is appropriate.

We fitted the more general model HTRP$(F_w, \lambda_{pl}(\cdot), H_g)$ which includes the possibility of a time trend given in the form of (3). Computations were done using REIDAR, and the resulting log-likelihoods are given in the log-likelihood cube of Figure 2. Looking at the numbers we conclude that the renewal distribution can be taken to be exponential, leaving us with the bottom face of the cube. Further, when comparing the front face to the back face there is reason to conclude that there is heterogeneity between the systems. The conclusions so far are thus in accordance with the conclusions of Proschan. Finally, comparing the left and the right faces of the cube, there is reason to believe that there is some time trend in the data. In fact, twice the log-likelihood difference from HHPP to HNHPP amounts to 5.24, giving a p-value of 0.022 assuming a chi-square distribution with one degree of freedom.

A graphical check of model fit was made by using TTT-plots based on the residual processes as described in Section 6. Figure 3 shows the TTT-plots for distribution of residuals for the models HHPP and HNHPP. In both cases the curve is close to the diagonal, indicating a reasonably good fit for both models.

TTT-plots for time trend are shown in Figure 4. The plot for the HHPP model indicates an increasing intensity in the corresponding residual processes (concave curve), while the residual processes of the HNHPP model seem to correspond much better to HPP. This conclusion was in fact supported by computation of the AD statistic for trend.

To check for a possible autocorrelation in the residual processes we computed $\rho(1)$ from (19) to be -0.13 and -0.09, respectively, for the models HHPP and HNHPP. Since the asymptotic standard deviation is 0.09, we cannot reasonably conclude that there is autocorrelation at lag 1, although the asymptotics may not be valid here since the number of failures of some of the systems are rather small.

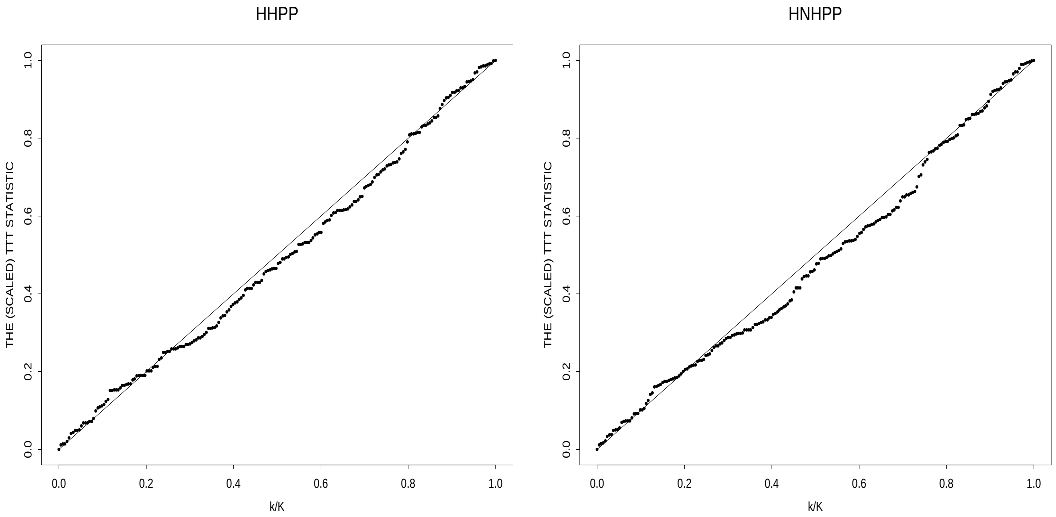

Figure 3. TTT-plots for distribution of residuals for models HHPP(ν, H_g) and HNHPP($\lambda_{pl}(\cdot), H_g$) for Proschan's data.

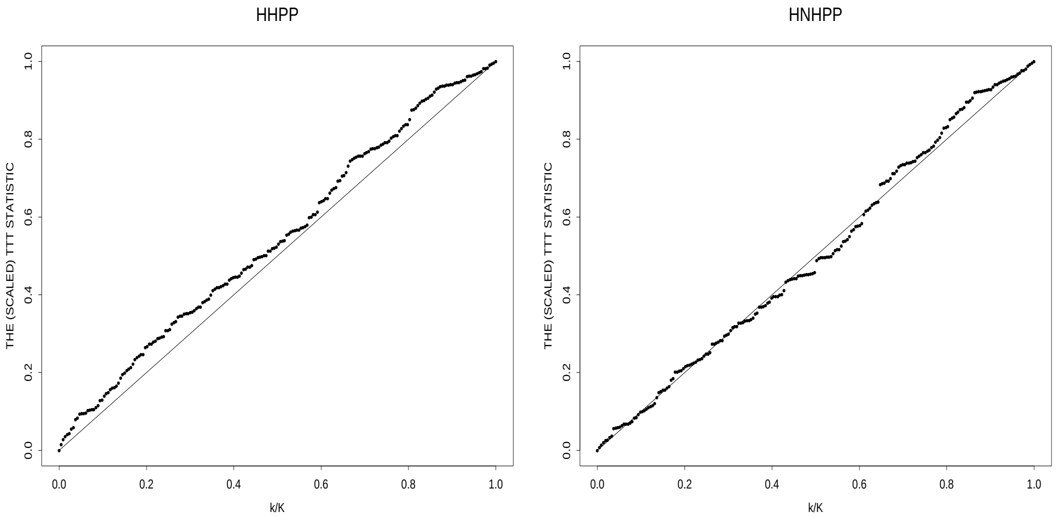

Figure 4. TTT-plots for time trend in residual processes for models HHPP(ν, H_g) and HNHPP($\lambda_{pl}(\cdot), H_g$) for Proschan's data.

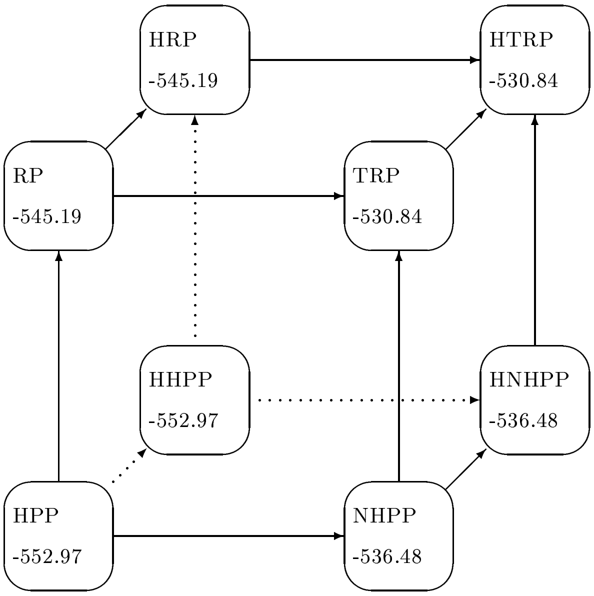

Figure 5. The log-likelihood cube of the HTRP($F_g, \lambda_{pl}(\cdot), H_g$) model for the tractor engine data

To conclude, when fitting the HTRP$(F_w, \lambda_{pl}(\cdot), H_g)$ model to Proschan's data, we find that a HNHPP$(\lambda_{pl}(\cdot), H_g)$ model seems to give a better fit than a HHPP(ν, H_g) model. Our analysis confirms, however, Proschan's conclusion of heterogeneity between systems and Poisson-behavior of individual systems.

7.2 Tractor Engine Data

These data are taken from Barlow and Davis (1977). They concern engine replacement times (in operating hours) for 22 tractors. The data are failure censored with a total number of 59 failures and the number of failures for each system varying from 1 to 6.

An HTRP$(F_g, \lambda_{pl}(\cdot), H_g)$ model was fitted to the data using the numerical routines of REIDAR. Note that this model was studied in Example 5.1. The resulting log-likelihood cube is shown in Figure 5. It is worth noting that none of the models HPP, NHPP, RP or TRP can be improved (with respect to increased likelihood) by introducing a non-degenerate gamma-distributed heterogeneity. To check the conclusion that $\hat{\delta} = 0$ for all the models of the back face we computed (16) for each of them and all the derivatives turned out to be negative.

Looking at the cube and starting from the HPP model we see that there is a large gain in log-likelihood when going either to a gamma renewal process (RP) or a power law non-homogeneous Poisson process (NHPP). A considerably larger log-likelihood is obtained, however, by further proceeding to the TRP model.

For illustration we computed residual processes and made TTT-plots for the three models NHPP, RP and TRP. Figure 6 shows plots for the distribution of residuals. For the TRP model there is a reasonably good fit, while there seems to be deviations from the diagonal for the NHPP and RP plots. The values of the AD-statistics are 2.86, 3.28 and 0.69, respectively, for the NHPP, RP and TRP models. P-values based on the AD-distribution are then 0.032, 0.020 and 0.69, and although these values may not be valid, they seem to confirm the conclusions of the plots.

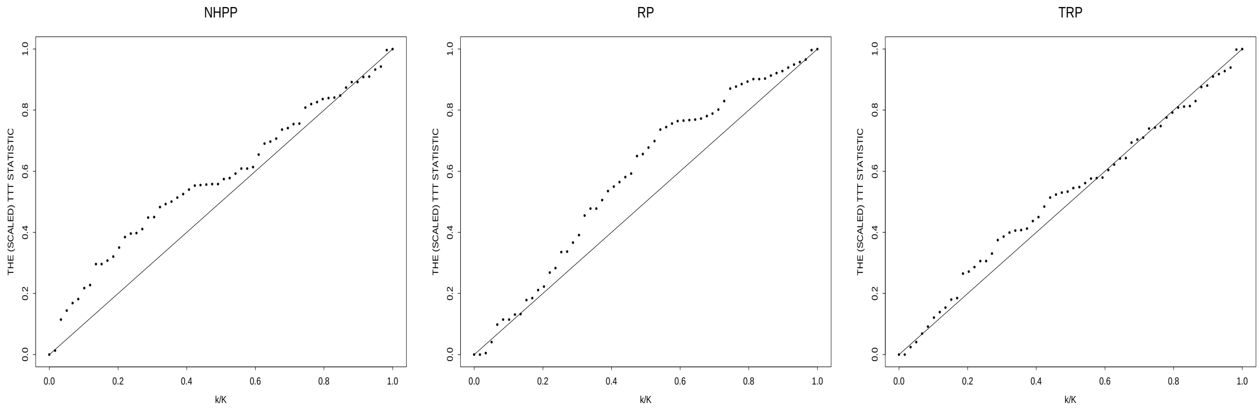

Figure 6. TTT-plots for distribution of residuals for models NHPP($\lambda_{pl}(\cdot)$), RP(F_g, ν), and TRP($F_g, \lambda_{pl}(\cdot)$) for tractor engine data.

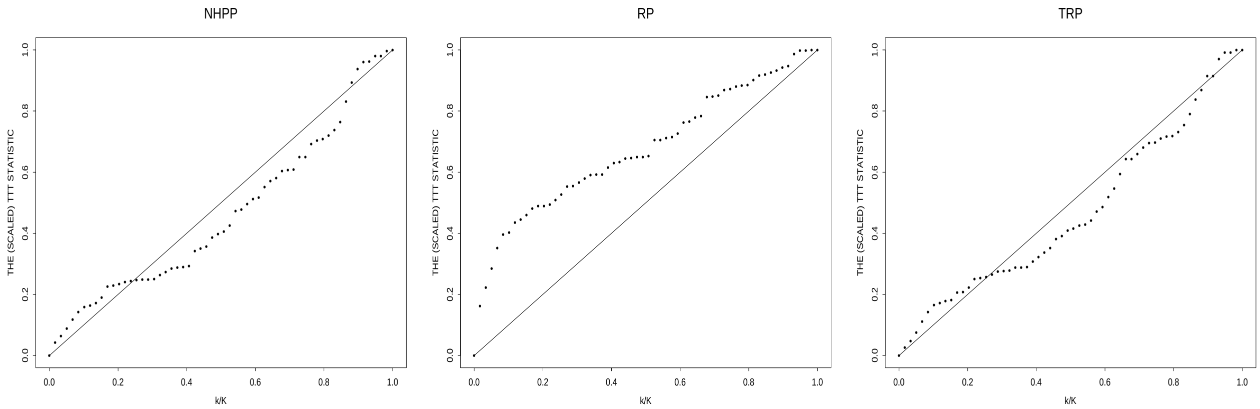

Figure 7. TTT-plots for time trend in residual processes for models NHPP($\lambda_{pl}(\cdot)$), RP(F_g, ν) and TRP($F_g, \lambda_{pl}(\cdot)$) for tractor engine data.

Table 2. The compressor data. There are n=41 failure times (in days), with time censoring at 7571 days.

1	4	305	330	651	856	996	1016	1155	1520	1597	1729
1758	1852	2070	2073	2093	2213	3197	3555	3558	3724	3768	4103
4124	4170	4270	4336	4416	4492	4534	4758	4762	5474	5573	5577
5715	6424	6692	6830	6999	(7571)						

From the TTT plots for time trend in Figure 7 we conclude that the RP seems inappropriate, while the plots for the NHPP and TRP models both show reasonably good fits. These conclusions were also supported by computation of the AD-statistics for trend.

To conclude, a TRP(F_g, λ_{pl}) seems to gives the best fit among the considered models. Thus our analyses indicate that there is a time trend present in the data, and that the interfailure times are not consistent with an NHPP model. A possible physical interpretation of the failure mechanisms based on this is that the replacement of the engine of the system leads to a renewal-type behavior, while the general aging of the rest of the system introduces an increased replacement frequency.

7.3 Compressor Data

The data contain 41 failure times of a gas compressor (Erlingsen 1989), with time censoring at time 7571 (days). The data are given in Table 2. We fitted the model TRP($F_w, \lambda_{pl}(\cdot)$) and its natural submodels, obtaining the log-likelihood values shown in Figure 8, which we now call the log-likelihood square.

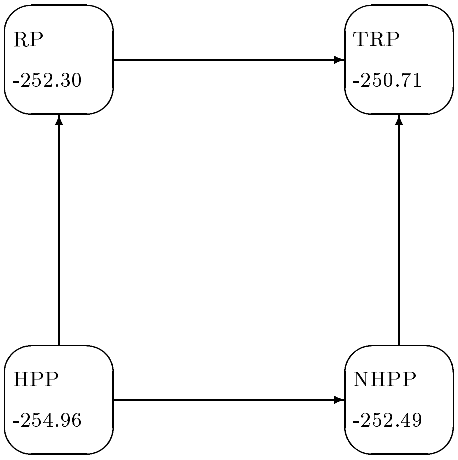

Figure 8. The log-likelihood square of the TRP($F_w, \lambda_{pl}(\cdot)$) model for the compressor data

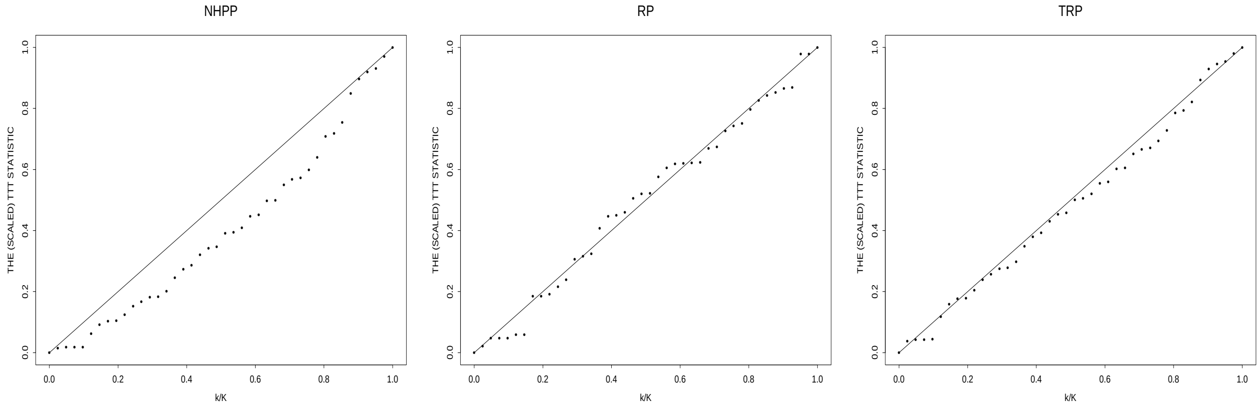

Figure 9. TTT-plots for distribution of residuals for models NHPP($\lambda_{pl}(\cdot)$), RP(F_w, ν) and TRP($F_w, \lambda_{pl}(\cdot)$) for compressor data.

Based on differences in log-likelihood and assuming that the usual chi-square approximations are valid, there is a rather clear evidence against the HPP model (P-values 0.026, 0.021 and 0.014, respectively, against the NHPP, RP and TRP models). There is also some evidence pointing to the TRP model as slightly better than the NHPP and RP models (P-values, respectively, 0.059 and 0.075).

We thus conclude that an HPP model is not appropriate as a model for these data, but that improved models may be found by allowing either a time trend (leading to an NHPP), or a non-exponential distribution of interfailure times (leading to RP), or possibly using a combination of these approaches (leading to a TRP).

In order to compare the fit of these three models we consider the resulting residual processes. Figures 9 and 10 show TTT-plots for, respectively, the distributions of the residuals and the time trends of the residual processes for the three models. Judged by these plots it seems that the NHPP is not appropriate as a model, due to the bad shape of the corresponding distribution plot. The corresponding plots for the RP and TRP models seem, however, to be satisfactorily concentrated near the diagonal, both for distribution and time trend.

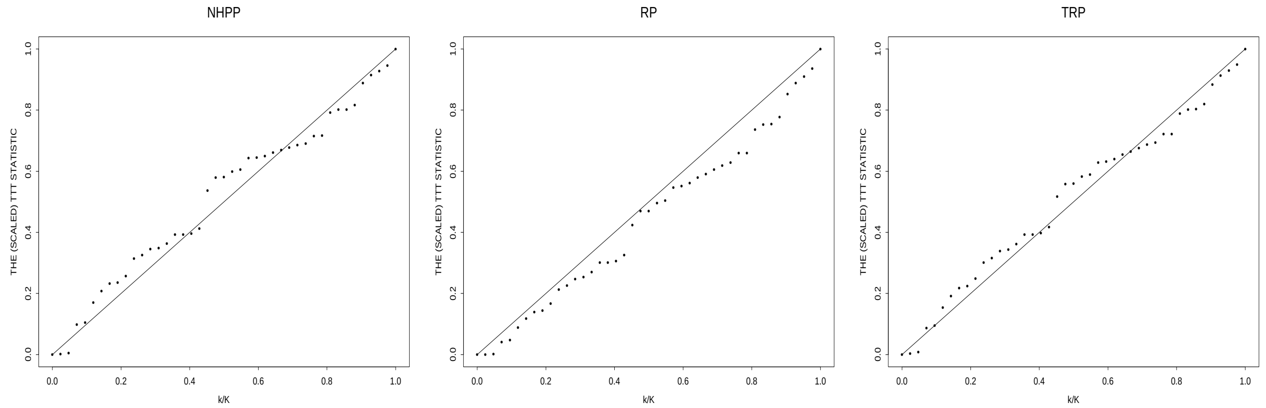

Figure 10. TTT-plots for time trend in residual processes for models NHPP($\lambda_{pl}(\cdot)$), RP(F_w, ν) and TRP($F_w, \lambda_{pl}(\cdot)$) for compressor data.

Table 3.Bootstrapping the Anderson-Darling statistics for the residuals of the compressor data. The table gives AD-statistics and corresponding P-values for testing the null hypothesis that the data come from the respective models.

TTT-statistic	*Model*	*AD statistic*	*P-value (bootstrap)*	*P-value (AD-distribution)*
	NHPP	*2.62*	*0.04*	*0.04*
Distribution	*RP*	*0.39*	*0.72*	*0.86*
	TRP	*0.28*	*0.88*	*0.95*
	NHPP	*0.80*	*0.17*	*0.48*
Trend	*RP*	*1.27*	*0.25*	*0.24*
	TRP	*0.59*	*0.44*	*0.66*

In order to confirm the above conclusions we computed the AD-statistics corresponding to each of the six TTT-plots. As noted in Section 6, the usual asymptotic distribution of the AD statistic may not be valid as an approximation for the distribution of the observed AD-statistics. We therefore used parametric bootstrapping to estimate the distribution of the AD statistics for each of the six plots. Then we used the bootstrap distributions to obtain estimated P-values. For each of the three models we generated 1000 bootstrap samples by substituting maximum likelihood estimates for the parameters and simulating as described in Section 5. The results are shown in Table 3. The first numeric column shows the computed values of the AD-statistics, while the second contains the P-values computed by bootstrapping. For comparison we have included a third column with P-values computed from the theoretical AD-distribution. The only reasonably significant P-value occurs for the distribution of residuals for the NHPP model. This is in accordance with the corresponding TTT-plot (Figure 9). None of the other P-values are suspiciously small, however, which is also in accordance with the TTT-plots.

The difference in bootstrapped P-values and P-values based on the AD-distribution are presumably due to the use of estimated parameters in the observed residuals. For precise results in this direction we refer to Stephens (1976).

The estimated autocorrelation functions at lag 1, using (19), are -0.06, -0.01 and -0.06, respectively, for the NHPP, RP and TRP models. P-values for testing the null hypothesis of independence, computed from the bootstrap samples, gave the values 0.49, 0.43, 0.53, respectively. Thus there is no indication of dependence in the residual processes. Such a conclusion seems also clear from the asymptotic standard deviation for the autocorrelation under independence, which by the formula in Section 6 is 0.16. Asymptotic results are believed to be valid here, and in fact this was confirmed by estimating the standard deviations of the autocorrelations by bootstrapping to be, respectively, 0.14, 0.16, 0.15 for the three models.

To conclude, both the RP(F_w, ν) and TRP(F_w, λ_{pl}) give reasonable fits to the compressor data. Since the RP is a simpler model, we might therefore recommend it to the TRP model.

8. CONCLUDING REMARKS

The TRP model extends in a conceptually simple way the NHPP and RP models, and useful TRP models are obtained by only moderate extensions of the original models. For example, using a one-parameter family for F, the extension from an NHPP model to a TRP model means adding only a single parameter. Moreover we obtain simple models which include heterogeneity, HTRP, by adding a single parameter to a TRP model. Although simple, the exteded models can be important for understanding the true underlying mechanisms and for model validation.

For example, in the previous section we found that the tractor engine data in Example 7.2 were not sufficiently explained by NHPP or RP models, while a TRP model (in fact a modulated power law process) indicated reasonably good fit. In the two other examples the selected models were, respectively, HNHPP (Proschan's data) and RP (compressor data). In these examples an important feature of our approach was to embed the classical Poisson- and renewal models in wider classes of TRP models, and thereby obtaining tests for model fit.

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REFERENCES

Aalen, O. O., and Husebye, E. (1991), "Statistical analysis of repeated events forming renewal processes". *Statistics in Medicine*, 10, 1227-1240.

Abramowitz, M. and Stegun, I. A. (1965), *Handbook of Mathematical Functions.* New York: Dover.

Andersen, P., Borgan, O., Gill, R., and Keiding, N. (1993), *Statistical Models Based on Counting Processes*, New York: Springer.

Anderson, T. W., and Darling, D. A. (1954), "A test of goodnees of fit", *Journal of the American Statistical Association*, 49, 765-769.

Ascher, H., and Feingold, H. (1984), *Repairable Systems – Modeling, Inference, Misconceptions and their Causes.* New York: Marcel Dekker.

Bain, L. J. and Wright, F. T. (1982), "The negative binomial process with applications to reliabiity", *Journal of Quality Technology*, 14, 60-66.

Barlow, R. E. and Campo, R. (1975), "Total time on test processes and applications to failure data analysis", in *Reliability and fault tree analysis*, eds. R. E. Barlow, J. B. Fussel and N. Singpurwalla, SIAM Philadelphia, pp. 451-481.

Barlow, R. E., and Davis, B. (1977), "Analysis of time between failures of repairable components", in *Nuclear Systems Reliability Engineering and Risk Assessment*, eds. J. B. Fussel and G. R. Burdick, SIAM Philadelphia, pp. 543-561.

Berman, M. (1981), "Inhomogeneous and modulated gamma processes", *Biometrika*, 68, 143-152.

Berman, M., and Turner, T. R. (1992), "Approximate point process likelihoods with GLIM", *Applied Statistics*, 41, 31-38.

Black, S. E., and Rigdon, S. E. (1996), "Statistical inference for a modulated power law process", *Journal of Quality Technology*, 28, 81-90.

Brémaud, P. (1981), *Point Processes and Queues: Martingale Dynamics*, New York: Springer.

Brown, M., and Proschan, F. (1983), "Imperfect repair", *Journal of Applied Probability*, 20, 851-859.

Calabria, R. and Pulcini, G. (1999), "On testing for repair effect and time trend in repairable mechanical units", *Communications in Statistics - Theory and Methods*, 28, 367-387.

Chernoff, H. (1954), "On the distribution of the likelihood ratio", *Annals of Mathematical Statistics*, 25, 573-578.

Cox, D. R. (1972), "The statistical analysis of dependencies in point processes", in *Stochastic Point Processes,* ed. P.A. Lewis, New York: Wiley, pp. 55-66.

Cox, D. R., and Snell, E. J. (1968), "A general definition of residuals (with discussion)", *Journal of the Royal Statistical Society,* Ser. B, 30, 248-275.

Crowder, M. J., Kimber, A. C., Smith, R. L., and Sweeting, T. J. (1991), *Statistical Analysis of Reliability Data*, London: Chapman and Hall.

Dorado, C., Hollander, M., and Sethuraman, J. (1997), "Nonparametric estimation for a general repair model", *Annals of Statistics*, 25, 1140-1160.

Efron, B. and Tibshirani, R. J. (1993), *An Introduction to the Bootstrap*, New York: Chapman and Hall.

Engelhardt, M. and Bain, L. J. (1987), "Statistical analysis of a compound power-law model for repairable systems", *IEEE Transactions on Reliability*, R-36, 392-396.

Erlingsen, S. E. (1989), "Using reliability data for optimizing maintenance", Master's Thesis, Norwegian Institute of Technology.

Follmann, D. A., and Goldberg, M. S. (1988), "Distinguishing heterogeneity from decreasing hazard rate", *Technometrics*, 30, 389-396.

Kay, R. (1977), "Proportional hazards regression models and the analysis of censored survival data", *Applied Statistics*, 26, 227-237.

Kvaløy, J. T., and Lindqvist, B. H. (1998), "TTT-based tests for trend in repairable systems data", *Reliability Engineering and System Safety*, 60, 13-28.

Lakey, M. J., and Rigdon, S. E. (1992), "The modulated power law process", *Proc. 45th Annual Quality Congress*, Milwaukee 1992, pp. 559-563.

Lawless, J. F. (1987), "Regression methods for Poisson process data", *Journal of the American Statistical Association*, 82, 808-815.

Lawless, J. F. (1995), "The analysis of recurrent events for multiple subjects", *Applied Statistics*, 44, 487-498.

Lawless, J. F., and Nadeau, C. (1995), "Some simple robust methods for the analysis of recurrent events", *Technometrics*, 37, 158-168.

Lawless, J. F. and Thiagarajah, K. (1996), "A point-process model incorporating renewals and time trends, with application to repairable systems", *Technometrics*, 38, 131-138.

Lehmann, E. L. (1999), *Elements of large-sample theory*, New York: Springer.

Maritz, J. S. and Lwin, T. (1989), *Empirical Bayes methods* (Second edition), London: Chapman and Hall.

Ng, E. T. M., and Cook, R. J. (1999), "Adjusted score tests of homogeneity for Poisson processes", *Journal of the American Statistical Association*, 94, 308-319.

Proschan, F. (1963), "Theoretical explanation of observed decreasing failure rates", *Technometrics*, 5, 375-383.

Ross, S. M. (1983), *Stochastic Processes*, New York: Wiley.

Stephens, M. A. (1976), "Asymptotic results for goodness-of-fit statistics with unknown parameters", *The Annals of Statistics*, 4, 357-369.

Svensson, Å. (1990), "Asymptotic estimation in counting processes with parametric intensities based on one realization", *Scandinavian Journal of Statistics*, 17, 23-33.

Xxx (1999), "REIDAR - an S-plus library for analysing data from repairable systems".